MATH 240 SPRING 2014

EXAM 1

Problem 2. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} + 2y = 3, \quad y(0) = 1.$$

This is a separable equation. General solution:

$$\frac{dy}{3-2y} = dx;$$
$$-(1/2)\ln|3-2y| = x + C;$$
$$\ln|3-2y| = -2x + C;$$
$$3 - 2y = Ce^{-2x};$$
$$y = (1/2)(3 + Ce^{-2x}).$$

Since $y(0) = 1$, $C = -1$. Thus

$$y = (1/2)(3 - e^{-2x}).$$

Problem 3. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} = 4x - y, \quad y(1) = 0.$$

This is a linear equation. Write

$$\frac{dy}{dx} + y = 4x.$$

Multiply by the integrating factor $\mu = e^x$:

$$e^x \frac{dy}{dx} + e^x y = 4xe^x;$$

$$\frac{d}{dx}(e^x y) = 4xe^x;$$

$$e^x y = \int 4xe^x dx = 4xe^x - 4\int e^x dx$$
$$= (4x-4)e^x + C,$$

where we used integration by parts. Since $y(1) = 0$, $C = 0$, and

$$e^x y = (4x-4)e^x.$$

Thus

$$y = 4x - 4.$$

Problem 4. (16 pts) Find all solutions of the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

This is a homogeneous equation, which can be re-written as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}.$$

Change of variables

$$v = y/x, \quad y = xv, \quad dy = xdv + vdx$$

leads to

$$x\frac{dv}{dx} + v = \frac{v - 4}{1 - v},$$

which is a separable equation. The general solution:

$$x\frac{dv}{dx} = \frac{v - 4}{1 - v} - v = \frac{v^2 - 4}{1 - v};$$

$$\frac{1 - v}{v^2 - 4}dv = \frac{dx}{x};$$

$$\int \frac{1 - v}{v^2 - 4}dv = \int \frac{dx}{x};$$

$$\int \frac{-1/4}{v - 2}dv + \int \frac{-3/4}{v + 2}dv = \int \frac{dx}{x};$$

$$\ln\left(|v - 2|^{-1/4}|v + 2|^{-3/4}\right) = \ln|x| + C;$$

$$\frac{1}{(v - 2)(v + 2)^3} = Cx^4;$$

$$x^4(v - 2)(v + 2)^3 = C.$$

Note that the last formula includes the singular solutions $v = 2$ and $v = -2$. Substituting $y = vx$ gives

$$(y - 2x)(y + 2x)^3 = C.$$

Problem 5. (16 pts) Find all solutions of the equation

$$\frac{dy}{dx} = \frac{4x - y}{x - y}.$$

This is another homogeneous equation, which can be solved similarly to the previous one. Another approach is to re-write it as

$$4x - y + (y - x)\frac{dy}{dx} = 0,$$

which is an exact equation because

$$\frac{\partial}{\partial y}(4x - y) = -1 = \frac{\partial}{\partial x}(y - x).$$

The general solution is of the form $F(x, y) = C$, where

$$\begin{cases} \dfrac{\partial F}{\partial x} = 4x - y, \\ \dfrac{\partial F}{\partial y} = y - x. \end{cases}$$

The first equation implies that $F(x, y)$ is of the form

$$F(x, y) = 2x^2 - xy + K(y);$$

the second equation implies that

$$\frac{dK}{dy} = y,$$

so $K(y) = y^2/2$ and $F(x, y) = 2x^2 - xy + y^2/2$. Thus the answer to the problem is

$$2x^2 - xy + y^2/2 = C,$$

or

$$4x^2 - 2xy + y^2 = C.$$

Problem 6. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^3 - 2xy}{x^2 - y^2}, \quad y(1) = 0.$$

Re-write the equation as

$$x^3 - 2xy + (y^2 - x^2)\frac{dy}{dx} = 0,$$

which is an exact equation because

$$\frac{\partial}{\partial y}(x^3 - 2xy) = -2x = \frac{\partial}{\partial x}(y^2 - x^2).$$

The general solution is of the form $F(x, y) = C$, where

$$\begin{cases} \dfrac{\partial F}{\partial x} = x^3 - 2xy, \\ \dfrac{\partial F}{\partial y} = y^2 - x^2. \end{cases}$$

The first equation implies that $F(x, y)$ is of the form

$$F(x, y) = x^4/4 - x^2 y + K(y);$$

the second equation implies that

$$\frac{dK}{dy} = y^2,$$

so $K(y) = y^3/3$ and $F(x, y) = x^4/4 - x^2 y + y^3/3$. Thus the general solution is

$$x^4/4 - x^2 y + y^3/3 = C,$$

or

$$3x^4 - 12x^2 y + 4y^3 = C.$$

Since $y(1) = 0$, $C = 3$, and

$$3x^4 - 12x^2 y + 4y^3 = 3.$$

Problem 7. (16 pts) Find all solutions of the equation

$$\frac{dy}{dx} - y + 2e^x y^2 = 0.$$

This is a Bernoulli equation. Re-write it as

$$\frac{dy}{dx} - y = -2e^x y^2$$

and observe that $y = 0$ is a singular solution. To find the general solution, divide by y^2:

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = -2e^x.$$

Change of variables

$$v = 1/y, \quad dv = -dy/y^2$$

leads to the equation

$$-\frac{dv}{dx} - v = -2e^x,$$

or

$$\frac{dv}{dx} + v = 2e^x,$$

which is linear. Multiply by the integrating factor $\mu = e^x$ to obtain

$$e^x\frac{dv}{dx} + e^x v = 2e^{2x};$$

$$\frac{d}{dx}(e^x v) = 2e^{2x};$$

$$e^x v = \int 2e^{2x} dx = e^{2x} + C;$$

$$v = e^x + Ce^{-x}.$$

Substituting back $y = 1/v$, one concludes that all solutions of the original Bernoulli equation are of the form

$$y = \frac{1}{e^x + Ce^{-x}} \quad \text{or} \quad y = 0.$$