Problem 2. (16 pts) Find and classify (as stable, semi-stable or unstable) the equilibria of the autonomous equation

$$\frac{dp}{dt} = p^3 - 4p^2 + 4p \quad \text{as} \quad t \to \infty.$$

If $p(t)$ is the solution of the equation, satisfying the initial condition $p(0) = 1$, what is the limit $\lim_{t\to\infty} p(t)$?

The equilibria are found as follows: since $p^3 - 4p^2 + 4p = p(p-2)^2 = 0$ when $p = 0$ or $p = 2$, we can conclude that $p = 0$ and $p = 2$ are the equilibrium solutions. Furthermore, since $p^3 - 4p^2 + 4p > 0$ for $0 < p < 2$ and for $p > 2$, while $p^3 - 4p^2 + 4p < 0$ for $p < 0$, the equilibrium at $p = 2$ is semi-stable as $t \to \infty$, and the equilibrium at $p = 0$ is unstable as $t \to \infty$. Finally, for the solution $p(t)$, satisfying the initial condition $p(0) = 1$, $p'(t) > 0$ and hence this solution is increasing towards the equilibrium $p = 2$. Thus $\lim_{t\to\infty} p(t) = 2$.

Problem 3. (16 pts) Consider the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(0) = 1.$$

Find the Picard iterations $y_1(x)$ and $y_2(x)$ for this problem, starting with $y_0 = 1$.

The Picard iteration formula for this problem reads:

$$y_{n+1}(x) = 1 + \int_0^x y_n(t)^2 dt.$$

For $n = 0$ we obtain

$$y_1(x) = 1 + \int_0^x y_0(t)^2 dt = 1 + \int_0^x dt = 1 + x.$$

For $n = 1$ we obtain

$$y_2(x) = 1 + \int_0^x y_1(t)^2 dt = 1 + \int_0^x (1+t)^2 dt = 1 + \frac{(1+x)^3 - 1}{3}$$

$$= 1 + x + x^2 + \frac{x^3}{3}.$$

Problem 4. (16 pts) Solve the initial value problem

$$y'' - 4y' + 4y = 2e^x, \quad y(0) = y'(0) = 0.$$

If y_p is some particular solution of the equation then the solution of the initial value problem is of the form $y = y_h + y_p$, where y_h is a solution of the homogeneous equation $y_h'' - 4y_h' + 4y_h = 0$. Thus

$$(D^2 - 4D + 4)y_h = 0;$$
$$(D - 2)^2 y_h = 0;$$
$$y_h = C_1 e^{2x} + C_2 x e^{2x},$$

where C_1 and C_2 are some constants. Now y_p can be found in the form $y_p = Ke^x$, where K is a constant and hence

$$y_p'' = y_p' = Ke^x.$$

Substituting in the equation, we obtain

$$Ke^x - 4Ke^x + 4Ke^x = 2e^x,$$

which implies $K = 2$. Thus $y_p = 2e^x$ and

$$y = C_1 e^{2x} + C_2 x e^{2x} + 2e^x.$$

Finally, since $y(0) = y'(0) = 0$, we have $C_1 + 2 = 2C_1 + C_2 + 2 = 0$, which implies $C_1 = -C_2 = -2$. Thus

$$y = -2e^{2x} + 2xe^{2x} + 2e^x.$$

Problem 5. (16 pts) Find the general solution of the equation

$$y'' + 2y' + 3y = 2\cos(x).$$

The general solution of the equation is of the form $y = y_h + y_p$, where y_p is some particular solution of the equation and y_h is the general solution of the homogeneous equation $y_h'' + 2y_h' + 3y_h = 0$. Thus

$$\begin{aligned}(D^2 + 2D + 3)y_h &= 0;\\ (D + 1 + i\sqrt{2})(D + 1 - i\sqrt{2})y_h &= 0;\\ y_h &= C_1 e^{(-1-i\sqrt{2})x} + C_2 e^{(-1+i\sqrt{2})x}\\ &= e^{-x}(K_1\cos(x\sqrt{2}) + K_2\sin(x\sqrt{2}))\\ &= Ae^{-x}\cos(x\sqrt{2} + \Theta),\end{aligned}$$

where $C_1, C_2, K_1, K_2, A, \Theta$ are arbitrary constants. Furthermore, since $2\cos(x) = \mathbf{Re}(2e^{ix})$, y_p can be found in the form $y_p = \mathbf{Re}(z)$, where z is a solution of the equation

$$z'' + 2z' + 3z = 2e^{ix}.$$

Set $z = Ke^{ix}$, where K is a constant, then $z' = iKe^{ix}$ and $z'' = -Ke^{ix}$. Therefore,

$$\begin{aligned}-Ke^{ix} + 2iKe^{ix} + 3Ke^{ix} &= 2e^{ix};\\ 2(1+i)K &= 2;\\ K = \frac{1}{1+i} &= \frac{1}{2} - \frac{1}{2}i = \frac{1}{\sqrt{2}}e^{-\pi i/4};\\ z = Ke^{ix} = \frac{1}{\sqrt{2}}e^{i(x-\pi/4)} &= \frac{\cos x + \sin x}{2} + \frac{\sin x - \cos x}{2}i;\\ y_p = \mathbf{Re}(z) = \frac{1}{\sqrt{2}}\cos(x - \pi/4) &= \frac{\cos x + \sin x}{2}.\end{aligned}$$

Thus

$$\begin{aligned}y &= Ae^{-x}\cos(x\sqrt{2} + \Theta) + \frac{1}{\sqrt{2}}\cos(x - \pi/4)\\ &= e^{-x}(K_1\cos(x\sqrt{2}) + K_2\sin(x\sqrt{2})) + \frac{\cos x + \sin x}{2}.\end{aligned}$$

Problem 6. (16 pts) Solve the initial value problem

$$\ddot{x} + 3\dot{x} + 2x = 3e^{-t}, \quad x(0) = \dot{x}(0) = 0.$$

If x_p is some particular solution of the equation then the solution of the initial value problem is of the form $x = x_h + x_p$, where x_h is a solution of the homogeneous equation $\ddot{x} + 3\dot{x} + 2x = 0$. Thus

$$(D^2 + 3D + 2)x_h = 0;$$
$$(D + 2)(D + 1)x_h = 0;$$
$$x_h = C_1 e^{-t} + C_2 e^{-2t},$$

where C_1 and C_2 are some constants. Since e^{-t} is a solution of the homogeneous equation, set $x_p = Kte^{-t}$, where K is a constant. Then

$$\dot{x_p} = K(1 - t)e^{-t}, \quad \ddot{x_p} = K(t - 2)e^{-t}.$$

Therefore,

$$K(t - 2)e^{-t} + 3K(1 - t)e^{-t} + 2Kte^{-t} = 3e^{-t},$$

which implies $K = 3$. Hence $x_p = 3te^{-t}$ and

$$x = C_1 e^{-t} + C_2 e^{-2t} + 3te^{-t}.$$

Finally, since $x(0) = \dot{x}(0) = 0$,

$$C_1 + C_2 = -C_1 - 2C_2 + 3 = 0,$$

which implies

$$C_1 = -C_2 = -3.$$

Thus

$$x = -3e^{-t} + 3e^{-2t} + 3te^{-t}.$$

Problem 7. (16 pts) Suppose an undamped spring-mass system has a mass of 5 g and resonates at a frequency of 13 Hz ($13\frac{\text{cycles}}{\text{sec}}$). A damping mechanism is then attached to the system, and it is observed that the free damped motion of the system is quasi-periodic with a frequency of 12 Hz. What is the spring constant of the system? What is the damping constant of the attached mechanism?

In the undamped scenario, the free motion is given by the equation

$$5\ddot{x} + kx = 0, \quad \text{or} \quad \left(D^2 + \frac{k}{5}\right) x = 0,$$

where k is the spring constant. Thus the *circular* frequency is

$$\omega_0 = \sqrt{\frac{k}{5}} = 2\pi \cdot 13 = 26\pi \left(\frac{\text{rad}}{\text{sec}}\right)$$

and we conclude that

$$k = 5 \cdot 26^2 \cdot \pi^2 = 3380\pi^2 \left(\frac{\text{g}}{\text{sec}^2}\right).$$

In the damped scenario, the free motion is given by the equation

$$5\ddot{x} + c\dot{x} + kx = 0, \quad \text{or} \quad (D - r_1)(D - r_2)x = 0,$$

where c is the damping constant, and

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 20k}}{10} = -r \pm \sqrt{r^2 - \omega_0^2},$$

where $r = c/10$. Since the motion is quasi-periodic (the system is under-damped),

$$r^2 - \omega_0^2 < 0,$$

and the circular frequency in this case is

$$\omega = \sqrt{\omega_0^2 - r^2} = 2\pi \cdot 12 = 24\pi \left(\frac{\text{rad}}{\text{sec}}\right).$$

It follows that

$$r^2 = \omega_0^2 - \omega^2 = 4\pi^2(13^2 - 12^2) = 4\pi^2(169 - 144) = 100\pi^2;$$

$$r = 10\pi;$$

$$c = 10r = 100\pi \left(\frac{\text{g}}{\text{sec}}\right).$$