Problem 2. (16 pts) Solve the following initial value problem, using Laplace transform:

$$\ddot{x} + 4\dot{x} + 5x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 2.$$

$$s^2 X - s - 2 + 4(sX - 1) + 5X = 0;$$
$$(s^2 + 4s + 5)X = s + 6;$$
$$X = \frac{s+6}{s^2+4s+5} = \frac{(s+2)+4}{(s+2)^2+1};$$
$$x = e^{-2t}(\cos(t) + 4\sin(t)).$$

Problem 3. (16 pts) Solve the following initial value problem, using Laplace transform:

$$\ddot{x} + 4\dot{x} + 3x = \delta(t-1), \quad x(0) = \dot{x}(0) = 0.$$

$$s^2X + 4sX + 3X = e^{-s};$$
$$X = \frac{e^{-s}}{s^2 + 4s + 3}.$$

Since

$$\frac{1}{s^2+4s+3} = \frac{1}{(s+1)(s+3)} = \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+3}\right)$$
$$= \mathcal{L}\left(\frac{e^{-t} - e^{-3t}}{2}\right),$$

$$x = u(t-1)\frac{e^{1-t} - e^{3-3t}}{2}.$$

Problem 4. (16 pts) Solve as a convolution integral (do not evaluate the integral)

$$\ddot{x} - 2\dot{x} + x = e^{t^2}, \quad x(0) = \dot{x}(0) = 0.$$

$$s^2 X - 2sX + X = \mathcal{L}\{e^{t^2}\};$$

$$X = \frac{\mathcal{L}\{e^{t^2}\}}{s^2 - 2s + 1} = \mathcal{L}\{te^t\}\mathcal{L}\{e^{t^2}\};$$

$$x = (te^t) * (e^{t^2}) = \int_0^t (t - \tau)e^{t-\tau+\tau^2} d\tau.$$

Problem 5. (16 pts) Find and classify (as stable, unstable or saddle) the equilibria of the system

$$\begin{cases} \dot{x} = x^2 - xy \\ \dot{y} = x + y + 2 \end{cases}$$

The equilibria are given by the system

$$\begin{cases} x^2 - xy = 0 \\ x + y + 2 = 0 \end{cases}$$

which has two solutions: either $x = 0, y = -2$ or $x = y = -1$.

Linearization at the equilibrium point $(0, -2)$ leads to the system

$$\begin{cases} \dot{u} = 2u \\ \dot{v} = u + v \end{cases}$$

where $u = x, v = y + 2$. Since both the trace and the determinant of the associated matrix

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

are positive, both poles of the Laplace transform are in the right half-plane and the equilibrium at $(0, -2)$ is unstable.

Linearization at the equilibrium point $(-1, -1)$ leads to the system

$$\begin{cases} \dot{u} = -u + v \\ \dot{v} = u + v \end{cases}$$

where $u = x + 1, v = y + 1$. Since the determinant of the associated matrix

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

is negative, the poles of the Laplace transform are on the opposite sides of the imaginary axis and the equilibrium at $(-1, -1)$ is a saddle point.

Problem 6. (16 pts) As we have seen, the general solution of the homogeneous Euler equation

$$x^2y'' + xy' - y = 0$$

is

$$y = Ax + \frac{B}{x},$$

where A and B are arbitrary constants. Use variation of parameters to solve the initial value problem

$$x^2y'' + xy' - y = 4x^3, \quad y(1) = y'(1) = 0.$$

First, re-write the inhomogeneous equation so that the leading coefficient is 1:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 4x.$$

Then, since $y_1 = x$ and $y_2 = 1/x$ are two independent solutions of the homogeneous equation, the general solution of the inhomogeneous equation is of the form

$$y = -x\int \frac{4x}{W(x,1/x)} \cdot \frac{1}{x}dx + \frac{1}{x}\int \frac{4x}{W(x,1/x)}xdx,$$

where

$$W(x,1/x) = x(1/x)' - (x)'(1/x) = -2/x.$$

Thus

$$y = x\int 2xdx - \frac{1}{x}\int 2x^3dx = \frac{x^3}{2} + Cx + \frac{D}{x},$$

where C and D are constants. Since $y(1) = y'(1) = 0$,

$$\frac{1}{2} + C + D = \frac{3}{2} + C - D = 0;$$

$$C = -1, D = \frac{1}{2};$$

$$y = \frac{x^3}{2} - x + \frac{1}{2x}.$$

Problem 7. (16 pts) Let

$$y(x) = \sum_{n=0}^{\infty} y_n x^n$$

be the solution of the initial value problem

$$(x^2+4)y'' + xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Determine the recurrence relation for the coefficients y_n. Find the coefficients y_0, y_1, y_2, y_3 and a lower bound for the radius of convergence of the power series $y(x)$.

Substitute the power series

$$y' = \sum_{n=1}^{\infty} n y_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) y_{n+1} x^n, \quad xy' = \sum_{n=0}^{\infty} n y_n x^n,$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) y_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) y_{n+2} x^n,$$

$$x^2 y'' = \sum_{n=0}^{\infty} n(n-1) y_n x^n$$

into the equation and group the matching powers of x together to obtain

$$\sum_{n=0}^{\infty} \left(n(n-1)y_n + 4(n+1)(n+2)y_{n+2} + n y_n - y_n \right) x^n = 0.$$

Thus the coefficients y_n satisfy the recurrence relation

$$4(n+1)(n+2)y_{n+2} + (n+1)(n-1)y_n = 0, \quad n = 0, 1, 2, \ldots$$

or

$$y_{n+2} = -\frac{n-1}{4(n+2)} y_n, \quad n = 0, 1, 2, \ldots$$

Since $y(0) = 1, y'(0) = 0$,

$$y_0 = 1, y_1 = 0, y_2 = \frac{1}{8} y_0 = \frac{1}{8}, y_3 = 0 y_1 = 0.$$

Finally, since the singular points of the equation in the complex plane

$$x^2 + 4 = 0 \Leftrightarrow x = \pm 2i$$

are at a distance of 2 from the origin, the radius of convergence of the series is at least 2.