Problem 1. (25 pts) Solve the initial value problem

$$\frac{dy}{dx} + y = e^x y^2, \quad y(0) = 1.$$

This is a Bernoulli equation; use the substitution

$$v = 1/y$$

to get

$$-\frac{dv}{dx} + v = e^x,$$

which is linear. Multiplying both sides by the integrating factor

$$\mu = e^{-x}$$

leads to

$$\frac{d}{dx}(-e^{-x}v) = 1,$$

hence

$$-e^{-x}v = x + C;$$
$$v = e^x(C - x);$$
$$y = \frac{e^{-x}}{C - x}.$$

Since $y(0) = 1$, $C = 1$, and

$$y = \frac{e^{-x}}{1 - x}.$$

Problem 2. (25 pts) Solve the initial value problem

$$\frac{dy}{dx} = \frac{8x - 3y}{3x - y}, \quad y(0) = 1.$$

Re-write the equation as

$$(3y - 8x)dx + (3x - y)dy = 0,$$

which is exact because

$$\frac{\partial(3y - 8x)}{\partial y} = \frac{\partial(3x - y)}{\partial x} = 3.$$

Hence the general solution is of the form $F(x, y) = C$, where

$$\begin{cases} \dfrac{\partial F}{\partial x} = 3y - 8x, \\ \dfrac{\partial F}{\partial y} = 3x - y, \end{cases} \qquad F(x, y) = 3xy - 4x^2 - y^2/2;$$

$$3xy - 4x^2 - y^2/2 = C \quad \text{or} \quad 8x^2 - 6xy + y^2 = C.$$

Since $y(0) = 1$, $C = 1$ and

$$8x^2 - 6xy + y^2 = 1$$

is the solution of the IVP.

Alternatively, one may treat the equation as homogeneous and use the substitution $y = xv$ which leads to

$$x\frac{dv}{dx} + v = \frac{8 - 3v}{3 - v},$$

$$x\frac{dv}{dx} = \frac{8 - 6v + v^2}{3 - v} = \frac{(v - 2)(v - 4)}{3 - v},$$

which is separable. Separating the variables, one gets:

$$\frac{(3 - v)dv}{(v - 2)(v - 4)} = \frac{dx}{x}.$$

Since

$$\int \frac{(3 - v)dv}{(v - 2)(v - 4)} = -\frac{1}{2}\int \left(\frac{1}{v - 2} + \frac{1}{v - 4}\right) dv = -\frac{1}{2}\ln|v^2 - 6v + 8| + C,$$

we obtain

$$\ln|v^2 - 6v + 8| = -2\ln|x| + C; \quad v^2 - 6v + 8 = \frac{C}{x^2};$$

$$y^2 - 6xy + 8 = C = 1$$

in view of the initial condition.

Problem 3. (25 pts) Solve the initial value problem

$$\ddot{x} - 3\dot{x} + 2x = e^t, \quad x(0) = \dot{x}(0) = 0.$$

Denote by $X(s)$ the Laplace transform of $x(t)$. Then

$$s^2X - 3sX + 2X = \frac{1}{s-1};$$

$$X = \frac{1}{(s-1)(s^2-3s+2)} = \frac{1}{(s-1)^2(s-2)}$$

$$= \frac{A}{s-2} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)};$$

$$A(s-1)^2 + B(s-2) + C(s-1)(s-2) = 1;$$

$$A = 1; \quad B = -1; \quad C = -1;$$

$$X = \frac{1}{s-2} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)};$$

$$x = e^{2t} - te^t - e^t.$$

Alternatively, write the solution as $x = x_h + x_p$ where x_h is a solution of the homogeneous equation

$$\ddot{x_h} - 3\dot{x_h} + 2x_h = 0,$$

and x_p is a particular solution of the original equation. Then

$$(D^2 - 3D + 2)x_h = 0; \quad (D-1)(D-2)x_h = 0; \quad x_h = Ae^t + Be^{2t}.$$

Therefore, x_p can be found in the form $x_p = Cte^t$; substitution yields

$$C(t+2)e^t - 3C(t+1)e^t + 2Cte^t = e^t; \quad C = -1; \quad x_p = -te^t.$$

Thus

$$x = Ae^t + Be^{2t} - te^t.$$

In view of the initial conditions

$$\begin{cases} A + B = 0; \\ A + 2B - 1 = 0; \end{cases} \quad \text{hence} \quad \begin{cases} A = -1; \\ B = 1; \end{cases}$$

$$x = e^{2t} - e^t - te^t.$$

Problem 4. (25 pts) Find and classify (as stable, unstable or saddle) the equilibria of the system

$$\begin{cases} \dot{x} = xy - 1 \\ \dot{y} = x - y \end{cases}$$

Find the equilibria:

$$\begin{cases} xy - 1 = 0; \\ x - y = 0; \end{cases} \quad \text{hence} \quad x = y = \pm 1$$

and the equilibrium points are $(1, 1)$ and $(-1, -1)$. Next, consider the matrix $M(x, y)$ associated with the linearized system at an equilibrium point (x, y):

$$M(x, y) = \begin{bmatrix} \frac{\partial(xy-1)}{\partial x} & \frac{\partial(xy-1)}{\partial y} \\ \frac{\partial(x-y)}{\partial x} & \frac{\partial(x-y)}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ 1 & -1 \end{bmatrix}.$$

In particular,

$$M(1, 1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};$$

since $\det M(1, 1) = -2 < 0$, the equilibrium at $(1, 1)$ is a saddle point.

Similarly,

$$M(-1, -1) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix};$$

since $\det M(1, 1) = 2 > 0$ and $\operatorname{trace} M(1, 1) = -2 < 0$, the equilibrium at $(-1, -1)$ is stable.

Problem 5. (25 pts) Solve the initial value problem

$$x^2y'' + 5xy' + 4y = 0, \quad y(1) = 0, y'(1) = 1.$$

This is an Euler equation; the associated indicial equation is

$$r^2 + 4r + 4 = 0$$

with the double root $r = -2$. Hence

$$y = x^{-2}(A + B\ln x),$$

where A and B are constants. In view of the initial conditions, $A = 0$ and $B = 1$, hence

$$y = \frac{\ln x}{x^2}.$$

Problem 6. (25 pts) It is known that the general solution of the homogeneous Euler equation

$$x^2y'' - 2xy' + 2y = 0$$

is

$$y = Ax + Bx^2,$$

where A and B are arbitrary constants. Use variation of parameters to solve the initial value problem

$$x^2y'' - 2xy' + 2y = x, \quad y(1) = y'(1) = 0.$$

Look for solution of the inhomogeneous equation in the form

$$y = xA(x) + x^2B(x),$$

where A and B are such that

$$xA'(x) + x^2B'(x) = 0.$$

Substitution in the equation yields

$$x^2(A'(x) + 2xB'(x)) = x.$$

Therefore,

$$\begin{cases} A' + xB' = 0; \\ A' + 2xB' = 1/x; \end{cases} \quad \begin{cases} A' = -1/x; \\ B' = 1/x^2; \end{cases} \quad \begin{cases} A = C - \ln x; \\ B = D - 1/x; \end{cases}$$

$$y = x(C - \ln x) + x^2(D - 1/x) = Cx + Dx^2 - x\ln x.$$

In view of the initial conditions,

$$\begin{cases} C + D = 0; \\ C + 2D - 1 = 0; \end{cases} \quad \begin{cases} C = -1; \\ D = 1; \end{cases}$$

$$y = x^2 - x - x\ln x.$$

Problem 7. (25 pts) Let

$$y(x) = \sum_{n=0}^{\infty} y_n x^n$$

be the solution of the initial value problem

$$(x^2+1)y'' + xy' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Determine the recurrence relation for the coefficients y_n. Find the coefficients y_0, y_1, y_2, y_3 and a lower bound for the radius of convergence of the power series $y(x)$.

First of all, since the singular points of the equation are at $x = \pm i$, the radius of convergence of the series is at least 1. Secondly, substituting the expansions

$$y' = \sum_{n=0}^{\infty}(n+1)y_{n+1}x^n, \quad xy' = \sum_{n=0}^{\infty} ny_n x^n,$$

$$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)y_{n+2}x^n, \quad x^2y'' = \sum_{n=0}^{\infty} n(n-1)y_n x^n$$

into the equation and grouping together the coefficients of the matching powers of x, one obtains

$$\sum_{n=0}^{\infty}(n(n-1)y_n + (n+2)(n+1)y_{n+2} + ny_n - 4y_n)x^n = 0.$$

Thus

$$(n+1)(n+2)y_{n+2} + (n^2-4)y_n = 0, \quad n = 0, 1, 2, \ldots.$$

or

$$y_{n+2} = \frac{2-n}{n+1}y_n, \quad n = 0, 1, 2, \ldots.$$

In view of the initial conditions,

$$y_0 = 1, \quad y_1 = 0, \quad y_2 = 2y_0 = 2, \quad y_3 = y_1/2 = 0.$$

Problem 8. (25 pts) Find and classify (as regular or irregular) the singular points of the equation

$$(x^3 - 2x^2 + x)y'' + (x-1)y' - y = 0.$$

Solve the indicial equation for every regular singular point you found.

First, find the singular points:

$$x^3 - 2x^2 + x = 0; \quad x(x-1)^2 = 0; \quad x = 0 \text{ and } x = 1.$$

Divide by the leading coefficient:

$$y'' + \frac{1}{x(x-1)}y' - \frac{1}{x(x-1)^2}y = 0.$$

At the singular point $x = 0$:

$$\lim_{x\to 0} x \cdot \frac{1}{x(x-1)} = -1;$$

$$\lim_{x\to 0} x^2 \cdot \frac{-1}{x(x-1)^2} = 0.$$

Since both limits exist, $x = 0$ is a regular singular point and the associated indicial equation is $r^2 - 2r = 0$ with the roots $r_1 = 2$, $r_2 = 0$.

At the singular point $x = 1$:

$$\lim_{x\to 1}(x-1) \cdot \frac{1}{x(x-1)} = 1;$$

$$\lim_{x\to 1}(x-1)^2 \cdot \frac{-1}{x(x-1)^2} = -1.$$

Since both limits exist, $x = 1$ is a regular singular point and the associated indicial equation is $r^2 - 1 = 0$ with the roots $r_1 = 1$, $r_2 = -1$.