MATH 240 SPRING 2015

EXAM 1

Problem 2. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} = 3y + 2x, \quad y(0) = 0.$$

This is a linear equation; multiply by the integrating factor $\mu = e^{-3x}$ to obtain

$$\begin{aligned} e^{-3x}\frac{dy}{dx} &= 3e^{-3x}y + 2xe^{-3x}; \\ e^{-3x}\frac{dy}{dx} - 3e^{-3x}y &= 2xe^{-3x}; \\ \frac{d(ye^{-3x})}{dx} &= 2xe^{-3x}. \end{aligned}$$

Integration by parts leads to

$$\begin{aligned} ye^{-3x} = \int 2xe^{-3x}dx = -\frac{2}{3}xe^{-3x} + \frac{2}{3}\int e^{-3x}dx \\ = -\frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} + C, \end{aligned}$$

hence the general solution is

$$y = Ce^{3x} - \frac{2}{3}x - \frac{2}{9}.$$

Since $y(0) = 0$, $C = 2/9$ and

$$y = \frac{2}{9}e^{3x} - \frac{2}{3}x - \frac{2}{9}.$$

Problem 3. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} = \frac{\sin x - y}{x - \cos y}, \quad y(0) = 0.$$

Re-write the equation as

$$(\sin x - y)dx + (\cos y - x)dy = 0.$$

Since

$$\frac{\partial(\sin x - y)}{\partial y} = -1 = \frac{\partial(\cos y - x)}{\partial x},$$

the above equation is exact. The general solution is of the form $F(x, y) = C$, where

$$\begin{cases} \dfrac{\partial F}{\partial x} = \sin x - y; \\ \dfrac{\partial F}{\partial y} = \cos y - x. \end{cases}$$

The first equation implies that F is of the form

$$F = -\cos x - xy + \varphi(y).$$

Substituting this formula into the second equation gives

$$-x + \varphi'(y) = \cos y - x,$$

hence

$$\varphi'(y) = \cos y, \quad \varphi(y) = \sin y, \quad F = \sin y - \cos x - xy$$

and the general solution is given implicitly by

$$\sin y - \cos x - xy = C.$$

Since $y(0) = 0$, $C = \sin 0 - \cos 0 = -1$, and

$$\sin y - \cos x - xy = -1.$$

Problem 4. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} = \frac{2x+4y}{3x+y}, \quad y(1) = 1.$$

This is a homogeneous equation; use the substitution $y = xv$ to rewrite it as a separable equation

$$x\frac{dv}{dx} + v = \frac{2+4v}{3+v};$$

$$x\frac{dv}{dx} = \frac{2+v-v^2}{3+v};$$

$$x\frac{dv}{dx} = -\frac{(v-2)(v+1)}{v+3}.$$

Note that the singular solutions $v = 2$ and $v = -1$ are irrelevant in view of the initial condition and separate the variables:

$$\int \frac{v+3}{(v-2)(v+1)}dv = -\int \frac{dx}{x}.$$

Using partial fractions to compute the integral on the left gives

$$\int \frac{v+3}{(v-2)(v+1)}dv = \int \left(\frac{5/3}{v-2} - \frac{2/3}{v+1}\right) dv$$

$$= \frac{5}{3}\ln|v-2| - \frac{2}{3}\ln|v+1| + C = \frac{1}{3}\ln\left|\frac{(v-2)^5}{(v+1)^2}\right| + C,$$

hence

$$\frac{1}{3}\ln\left|\frac{(v-2)^5}{(v+1)^2}\right| = -\ln|x| + C;$$

$$\ln\left|\frac{x^3(v-2)^5}{(v+1)^2}\right| = C;$$

$$\frac{x^3(v-2)^5}{(v+1)^2} = C;$$

$$\frac{(y-2x)^5}{(y+x)^2} = C;$$

where the reverse substitution $v = y/x$ was used to arrive at the last formula. Finally, since $y(1) = 1$, $C = -1/4$, and the solution of the initial value problem is given implicitly by

$$\frac{(y-2x)^5}{(y+x)^2} = -\frac{1}{4} \quad \text{or} \quad 4(y-2x)^5 + (y+x)^2 = 0.$$

Problem 5. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} = 2y + e^x y^2, \quad y(0) = 1.$$

This is a Bernoulli equation; in view of the initial condition one can ignore the singular solution $y = 0$ and write

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{2}{y} + e^x.$$

Use the substitution $y = 1/v$ to obtain

$$-\frac{dv}{dx} = 2v + e^x \quad \text{or} \quad \frac{dv}{dx} + 2v = e^x,$$

which is a linear equation. Multiply by the integrating factor $\mu = e^{2x}$ to obtain

$$\begin{aligned} e^{2x}\frac{dv}{dx} + 2e^{2x}v &= e^{3x}; \\ \frac{d(e^{2x}v)}{x} &= e^{3x}; \\ e^{2x}v &= \frac{1}{3}e^{3x} + C; \\ e^{2x}v &= \frac{e^{3x} + C}{3}; \\ v &= \frac{e^x + Ce^{-2x}}{3}; \\ y = \frac{1}{v} &= \frac{3}{e^x + Ce^{-2x}}. \end{aligned}$$

Since $y(0) = 1$, $C = 2$, and

$$y = \frac{3}{e^x + 2e^{-2x}}.$$

Problem 6. (16 pts) Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 2xy}{y^2 - x^2}, \quad y(1) = 2.$$

Re-write the equation as

$$(3x^2 + 2xy)dx + (x^2 - y^2)dy = 0.$$

Since

$$\frac{\partial(3x^2 + 2xy)}{\partial y} = 2x = \frac{\partial(x^2 - y^2)}{\partial x},$$

the above equation is exact. The general solution is of the form $F(x, y) = C$, where

$$\begin{cases} \dfrac{\partial F}{\partial x} = 3x^2 + 2xy; \\ \dfrac{\partial F}{\partial y} = x^2 - y^2. \end{cases}$$

The first equation implies that F is of the form

$$F = x^3 + x^2 y + \varphi(y).$$

Substituting this formula into the second equation gives

$$x^2 + \varphi'(y) = x^2 - y^2,$$

hence

$$\varphi'(y) = -y^2, \quad \varphi(y) = -\frac{y^3}{3}, \quad F = x^3 + x^2 y - \frac{1}{3}y^3,$$

and the general solution is given implicitly by

$$x^3 + x^2 y - \frac{1}{3}y^3 = C.$$

Since $y(1) = 2$, $C = 1/3$, and the solution of the initial value problem is given implicitly by

$$x^3 + x^2 y - \frac{1}{3}y^3 = \frac{1}{3} \quad \text{or} \quad 3x^3 + 3x^2 y - y^3 = 1.$$

Problem 7. (16 pts) Find and classify (as stable, unstable or semi-stable when $t \to \infty$) all the equilibria of the equation

$$\frac{dy}{dx} = 6y - y^2 - y^3.$$

Since

$$6y - y^2 - y^3 = -y(y^2 + y - 6) = -y(y+3)(y-2),$$

the equlibrium solutions are

$$y = 0, \quad y = -3, \quad y = 2.$$

In view of the signs of the right hand side of the equation, the slope field is increasing in the strips $-\infty < y < -3$, $0 < y < 2$ and decreasing in the strips $-3 < y < 0$, $2 < y < \infty$. Hence, as $t \to \infty$, the equilibria $y = 2$ and $y = -3$ are stable and the equilibrium $y = 0$ is unstable.