MATH 240 SPRING 2015

EXAM 2

Problem 2. (16 pts) Consider the initial value problem

$$\frac{dy}{dx} = x + y^2, \quad y(0) = 0.$$

Find the Picard iterations $y_1(x)$ and $y_2(x)$ for this problem, starting with $y_0 = 0$.

Use Picard's formula

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} (t + y_n(t)^2)dt$$

with $x_0 = y_0 = 0$ to find

$$y_1(x) = \int_0^x (t + y_0(t)^2)dt = \int_0^x tdt = \frac{x^2}{2};$$

$$y_2(x) = \int_0^x (t + y_1(t)^2)dt = \int_0^x (t + \frac{t^4}{4})dt = \frac{x^2}{2} + \frac{x^5}{20}.$$

Problem 3. (16 pts) Solve the initial value problem

$$y'' - 6y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Using the differential operator notation

$$D = \frac{d}{dx},$$

re-write the equation as

$$(D^2 - 6D + 9)y = 0;$$
$$(D - 3)^2 y = 0;$$
$$y = (C_1 + C_2 x)e^{3x},$$

where C_1 and C_2 are constants. In view of the initial conditions,

$$C_1 = 1 \quad \text{and} \quad 3C_1 + C_2 = 0.$$

Thus $C_2 = -3$ and

$$y = (1 - 3x)e^{3x}.$$

Problem 4. (16 pts) Find the general *real* solution of the equation

$$y'' + 2y' + 10y = 0.$$

Re-write the equation as

$$(D^2 + 2D + 10)y = 0;$$

$$(D + 1 + 3i)(D + 1 - 3i)y = 0;$$

$$y = C_1 e^{-1-3i}x + C_2 e^{-1+3i}x = e^{-x}(C_1 e^{-3xi} + C_2 e^{3xi}),$$

where C_1 and C_2 are arbitrary complex constants. Since

$$e^{\pm 3xi} = \cos(3x) \pm i \sin(3x),$$

the general complex solution can be written as

$$y = e^{-x}(K_1 \cos(3x) + K_2 \sin(3x)),$$

where $K_1 = C_1 + C_2$ and $K_2 = (C_2 - C_1)i$ are again arbitrary complex constants. The general *real* solution is given by the formula above where K_1 and K_2 are arbitrary *real* constants. Furthermore, converting $(K_1, -K_2)$ into polar coordinates,

$$K_1 - K_2 i = Ae^{i\Theta},$$

the general real solution can also be written as

$$y = e^{-x}\Re[(K_1 - K_2 i)e^{3xi}] = Ae^{-x}\cos(3x + \Theta),$$

where A and Θ are again arbitrary real constants.

Problem 5. (16 pts) Solve the initial value problem

$$y'' + 5y' - 6y = 3e^x, \quad y(0) = y'(0) = 0.$$

The general solution of this linear equation is of the form $y = y_h + y_p$, where y_p is some particular solutuion and y_h is the general solution of the homogeneous equation

$$y_h'' + 5y_h' - 6y_h = 0.$$

Thus

$$(D^2 + 5D - 6)y_h = 0;$$
$$(D + 6)(D - 1)y_h = 0;$$
$$y_h = C_1 e^{-6x} + C_2 e^x,$$

where C_1 and C_2 are arbitrary constants. Since $y = e^x$ is a particular solution of the homogeneous equation, one should look for y_p of the form $y_p = Cxe^x$, where C is a constant. Then

$$y_p' = C(x + 1)e^x, \quad y_p'' = C(x + 2)e^x,$$

hence the equation

$$y_p'' + 5y_p' - 6y_p = 3e^x$$

reduces to

$$C(x + 2)e^x + 5C(x + 1)e^x - 6Cxe^x = 3e^x, \quad 7C = 3, \quad C = \frac{3}{7}.$$

Thus $y_p = \frac{3}{7}xe^x$ and the general solution of the original equation is

$$y = y_h + y_p = C_1 e^{-6x} + C_2 e^x + \frac{3}{7}xe^x.$$

In view of the initial conditions,

$$C_1 + C_2 = 0 \quad \text{and} \quad -6C_1 + C_2 + \frac{3}{7} = 0.$$

Hence $C_1 = -C_2 = \frac{3}{49}$ and

$$y = \frac{3}{49}e^{-6x} - \frac{3}{49}e^x + \frac{3}{7}xe^x.$$

Problem 6. (16 pts) Solve the initial value problem

$$\ddot{x} + 5\dot{x} + 6x = 30cos(6t), \quad x(0) = \dot{x}(0) = 0.$$

The general solution of the equation is of the form $x = x_h + x_p$, where x_p is some particular solutuion and x_h is the general solution of the homogeneous equation

$$\ddot{x}_h + 5\dot{x}_h + 6x_h = 0.$$

Thus

$$(D^2 + 5D + 6)x_h = 0;$$
$$(D + 2)(D + 3)x_h = 0;$$
$$x_h = C_1e^{-2t} + C_2e^{-3t},$$

where C_1 and C_2 are arbitrary constants. To find x_p, observe that $30\cos(6t) = \Re(30e^{6ti})$ and consider the complex equation

$$\ddot{z} + 5\dot{z} + 6z = 30e^{6ti}.$$

This equation has a particular solution of the form $z = Ce^{6ti}$, where C is a complex constant. Since

$$\dot{z} = 6iCe^{6ti}, \quad \ddot{z} = -36Ce^{ti},$$

the equation for z reduces to

$$-36Ce^{ti} + 30iCe^{6ti} + 6Ce^{6ti} = 30e^{6ti}, \quad -30C(1 - i) = 30,$$
$$C = \frac{-1}{1 - i} = -\frac{1 + i}{2}, \quad z = -\frac{1 + i}{2}e^{6ti}.$$

Now $x_p = \Re(z)$, hence

$$x_p = \Re\left[-\frac{1 + i}{2}e^{6ti}\right] = -\frac{1}{2}\Re[(1 + i)(\cos(6t) + i\sin(6t)]$$
$$= \frac{\sin(6t) - \cos(6t)}{2};$$
$$x = x_h + x_p = C_1e^{-2t} + C_2e^{-3t} + \frac{\sin(6t) - \cos(6t)}{2}.$$

In view of the initial conditions,

$$C_1 + C_2 - \frac{1}{2} = 0 \quad \text{and} \quad -2C_1 - 3C_2 + 3 = 0.$$

Hence $C_1 = -\frac{3}{2}$, $C_2 = 2$ and

$$x = -\frac{3}{2}e^{-2t} + 2e^{-3t} + \frac{\sin(6t) - \cos(6t)}{2}.$$

Problem 7. (16 pts) Suppose an undamped spring-mass system has a mass of 60 g and resonates at a frequency of 0.5 Hz (that is, $0.5\frac{\text{cycles}}{\text{sec}}$). A damping mechanism is then attached to the system, and it is observed that the free damped motion of the system is quasi-periodic with a period of 2.5 sec. What is the spring constant of the system? What is the damping constant of the attached mechanism?

The resonating *circular* frequency ω_0 of the undamped system is given by

$$\omega_0 = 2\pi \cdot 0.5 = \pi(\frac{\text{rad}}{\text{sec}}).$$

On the other hand,

$$\omega_0 = \sqrt{\frac{k}{m}},$$

where $m = 60\,\text{g}$ is the mass and k is the spring constant (in $\frac{\text{g}}{\text{sec}^2}$). Thus

$$k = m\omega_0^2 = 60\pi^2 \left(\frac{\text{g}}{\text{sec}^2}\right).$$

Furthermore, since the (quasi-)period of damped motion is 2.5 sec, the circular (quasi-)frequency ω is

$$\omega = \frac{2\pi}{2.5} = 0.8\pi \left(\frac{\text{rad}}{\text{sec}}\right).$$

But the frequency of the damped motion is related to the resonating frequency by

$$\omega_0^2 - \omega^2 = \frac{c^2}{4m^2},$$

where c is the damping constant (in $\frac{\text{g}}{\text{sec}}$). Thus

$$c = 2m\sqrt{\omega_0^2 - \omega^2} = 120\sqrt{\pi^2 - 0.64\pi^2} = 72\pi \left(\frac{\text{g}}{\text{sec}}\right).$$