MATH 240 SPRING 2015

EXAM 3

Problem 2. (16 pts) Solve the following initial value problem, using Laplace transform:

$$\ddot{x} + 4\dot{x} + 4x = \delta(t-2), \quad x(0) = \dot{x}(0) = 0.$$

Denote by $X(s)$ the Laplace transform of $x(t)$. In view of the initial conditions,

$$s^2 X + 4sX + 4X = e^{-2s},$$

$$X = \frac{e^{-2s}}{s^2+4s+4} = \frac{e^{-2s}}{(s+2)^2} = e^{-2s}\mathcal{L}\{te^{-2t}\},$$

$$x(t) = u(t-2)e^{-2(t-2)}(t-2),$$

where $u(t)$ is the step function.

Problem 3. (16 pts) Solve the following system:

$$\begin{cases} \dot{x} = x - y, & x(0) = 1, \\ \dot{y} = 5x - y, & y(0) = 1. \end{cases}$$

Denote by $X(s)$ and $Y(s)$ the Laplace transforms of $x(t)$ and $y(t)$, respectively. In view of the initial conditions,

$$\begin{cases} sX - 1 = X - Y, \\ sY - 1 = 5X - Y, \end{cases} \quad \begin{cases} (s-1)X + Y = 1, \\ 5X - (s+1)Y = -1, \end{cases}$$

$$\begin{cases} 1 - (s-1)X = Y, \\ (s^2+4)X = s, \end{cases} \quad \begin{cases} Y = \dfrac{s+4}{s^2+4}, \\ X = \dfrac{s}{s^2+4}, \end{cases}$$

$$\begin{cases} x = \cos(2t), \\ y = \cos(2t) + 2\sin(2t). \end{cases}$$

Problem 4. (16 pts) Solve as a convolution integral (do not evaluate the integral)

$$\ddot{x} - 3\dot{x} + 2x = e^{\sin t}, \quad x(0) = \dot{x}(0) = 0.$$

Denote by $X(s)$ the Laplace transform of $x(t)$. In view of the initial conditions,

$$s^2X - 3sX + 2X = \mathcal{L}\{e^{\sin t}\}, \quad X = \frac{\mathcal{L}\{e^{\sin t}\}}{s^2 - 3s + 2},$$

$$x = e^{\sin t} * \mathcal{L}^{-1}\{\frac{1}{s^2 - 3s + 2}\} = e^{\sin t} * \mathcal{L}^{-1}\{\frac{1}{s-2} - \frac{1}{s-1}\}$$

$$= e^{\sin t} * (e^{2t} - e^t) = \int_0^t e^{\sin \tau}(e^{2t-2\tau} - e^{t-\tau})d\tau = \int_0^t e^{t-2\tau+\sin \tau}(e^t - e^\tau)d\tau.$$

Problem 5. (16 pts) Find and classify (as stable, unstable or saddle) the equilibria of the system

$$\begin{cases} \dot{x} = x^2 - 2xy - 3y^2 \\ \dot{y} = x - y - 1 \end{cases}$$

To find the equilibria, solve the system

$$\begin{cases} x^2 - 2xy - 3y^2 = 0, \\ x - y - 1 = 0, \end{cases} \quad \begin{cases} (x-y)^2 = 4y^2, \\ x - y = 1, \end{cases} \quad \begin{cases} x = 1 \pm \dfrac{1}{2}, \\ y = \pm \dfrac{1}{2}. \end{cases}$$

Thus the equilibrium points are $(3/2, 1/2)$ and $(1/2, -1/2)$.

Next, consider the matrix M associated with the linearized system:

$$M = \begin{bmatrix} 2x - 2y & -2x - 6y \\ 1 & -1 \end{bmatrix}.$$

In particular, $\text{trace}(M) = 2x - 2y - 1$ and $\det(M) = 8y$. At the equilibrium point $(\frac{3}{2}, \frac{1}{2})$ $\det(M) > 0$ and $\text{trace}(M) > 0$ hence it's an unstable equilibrium. At the equilibrium point $(\frac{1}{2}, -\frac{1}{2})$ $\det(M) < 0$ hence it's a saddle point.

Problem 6. (16 pts) It is known that the general solution of the homogeneous Euler equation

$$x^2y'' - 3xy' + 3y = 0$$

is $y = Ax + Bx^3$, where A and B are arbitrary constants.

Use variation of parameters to find the general solution of the inhomogeneous equation

$$x^2y'' - 3xy' + 3y = x.$$

The general solution of the inhomogeneous equation can be found in the form

$$y = A(x)x + B(x)x^3,$$

where

$$A'(x) = -\frac{f(x)x^3}{W(x,x^3)}, \quad B'(x) = \frac{f(x)x}{W(x,x^3)}, \quad f(x) = \frac{x}{x^2} = \frac{1}{x},$$

and $W(x,x^3)$ is the Wronskian

$$W(x,x^3) = x(x^3)' - (x)'x^3 = 2x^3.$$

Thus

$$A(x) = -\int \frac{dx}{2x} = -\frac{\ln|x|}{2} + C, \quad B(x) = \int \frac{dx}{2x^3} = -\frac{1}{4x^2} + D,$$

and

$$y = -\frac{x\ln|x|}{2} + (C - \frac{1}{4})x + Dx^3 = -\frac{x\ln|x|}{2} + Cx + Dx^3,$$

where C and D are arbitrary constants.

Problem 7. (16 pts) Let

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

be the solution of the initial value problem

$$(x^2+2)y'' + 6xy' + 6y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

Determine the recurrence relation for the coefficients c_n. Find the coefficients c_0, c_1, c_2, c_3 and a lower bound for the radius of convergence of the power series $y(x)$.

Since

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n,$$

$$(x^2+2)y'' = x^2y''+2y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)c_{n+2}x^n$$

$$= \sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)c_{n+2}x^n$$

$$= \sum_{n=0}^{\infty} (n(n-1)c_n + 2(n+2)(n+1)c_{n+2})x^n.$$

Similarly,

$$6xy' + 6y = \sum_{n=0}^{\infty} (6nc_n + 6c_n)x^n.$$

Since

$$(x^2+2)y'' + 6xy' + 6y = 0,$$

$$n(n-1)c_n + 2(n+2)(n+1)c_{n+2} + 6nc_n + 6c_n = 0, \quad n = 0, 1, 2, \ldots$$

which leads to the recurrence relation

$$c_{n+2} = -\frac{n^2+5n+6}{2(n+1)(n+2)}c_n = -\frac{n+3}{2n+2}c_n, \quad n = 0, 1, 2, \ldots$$

Furthermore, in view of the initial conditions, $c_0 = 1$, and $c_1 = 2$. The recurrence relation for $n = 0$ and $n = 1$ gives

$$c_2 = -\frac{3}{2}c_0 = -\frac{3}{2}, \quad c_3 = -c_1 = -2.$$

Finally, since the roots of x^2+2 are $x = \pm i\sqrt{2}$, the radius of convergence is at least $|i\sqrt{2}| = \sqrt{2}$.