Final Exam - Woo Hoo!!!
Math 240 Spring 2017
May 10, 2017

Name: ______________________________

Time of recitation: ______________________________

Name or initials of recitation instructor: ____________________

You may not use any type of calculator whatsoever. (Cell phones off and away!) You are not allowed to have any other notes, and the test is closed book. Use the backs of pages for scrapwork, and if you write anything on the back of a page which you want to be graded, then you should indicate that fact (on the front). Except for the three cheat sheets at the end of the test, do not unstaple or remove pages from the exam.

By taking this exam you are agreeing to abide by KSU's Academic Integrity Policy.

Simple or standard simplifications should be made. You must **show your work** for every problem, and in order to get credit or partial credit, your work must make sense!

GOOD LUCK!!! MAY THE FORCE BE WITH YOU, YOUNG JEDIS!!!

Problem	Possible	Score	Problem	Possible	Score
1	20		6	12	
2	15		7	18	
3	15		8	20	
4	20		9	20	
5	15		10	20	
Total	85			90	

1. Find the general solution of:

$$y'(t) = -\frac{2}{t}y(t) + \frac{1}{t}e^{3t}$$

for $t > 0$. (Do **not** try to use the Laplace transform.)

2. Write down the form of the best guess to use for the method of undetermined coefficients for the following ODE:

$$y''(t) + 9y'(t) + 14y(t) = 7\sin 3t - 8e^{-2t} + 9e^{2t} - 10te^{-7t} + 11t^3 \,.$$

You do not need to find the coefficients!

3. Suppose you are given the following initial value problem:

$$y''(t) = 2t^2 \sin(1 + y'(t)^2 + y(t)^5), \quad y(-2) = -3,\ y'(-2) = -5\ .$$

Do the following:

(a) Convert the problem above into a first order system.

(b) Next, set up the standard Euler iteration with a stepsize of 0.1 (Explicitly give y_0 and z_0 and give y_{n+1} and z_{n+1} as functions of y_n, z_n, and n.)

(c) Finally, in terms of this scheme, what would you use to approximate $y(3)$?

4. Solve using the Laplace Transform:

$$x''(t) - 7x'(t) + 10x(t) = 3\delta(t-6) - 5\delta(t-8)$$

$$x(0) = x'(0) = 0\ .$$

5. Find $y(t)$ by **using** the Laplace transform. (You don't need to find $x(t)$.)

$$
\begin{aligned}
x'(t) &= x(t) + 3y(t) & \qquad x(0) &= 1 \\
y'(t) &= -3x(t) + y(t) & \qquad y(0) &= 2
\end{aligned}
$$

6. Solve:

$$y''(x) + 2xy'(x) + y(x) = 0 \qquad y(0) = 0,\ y'(0) = 3$$

by using a power series centered at zero. Be sure to write down a recurrence relation, and find the first three nonzero terms in the expansion.

7. Consider the differential equation:

$$x^3(x+1)y'' + 3x^2y' + y = 0.$$

Zero and -1 are obviously singular points.
Do the following:

(a) Classify these points (0 and -1) as regular or irregular.

(b) At any regular singular points, find the indicial equation and the roots of the indicial equation.

(c) At any regular singular point, state where the series solution corresponding to the larger root would converge. **Do not try to find the solution! You don't have time!**

8. The equation:

$$x^2y''(x) + (x^2 + x)y'(x) - 4y(x) = 0$$

has a regular singular point at zero.

For this equation, $\alpha_0 = 1$ and $\beta_0 = -4$, and so the indicial equation is $s^2 - 4 = 0$. Find a series solution to the ODE above which corresponds to the larger root of the indicial equation.

9. (a) Write down the definition of the Laplace Transform, $\mathcal{L}[f]$.

(b) Find all solutions of:

$$y''(x) + 4y(x) = 0 \qquad y(0) = -1, \;\; y(\pi/2) = 1 \,.$$

(c) Using the fact that x^{-1} and x^{-2} are linearly independent solutions of

$$x^2 y''(x) + 4xy'(x) + 2y(x) = 0 \,,$$

find the general solution of the following problem for $x > 0$:

$$x^2 y''(x) + 4xy'(x) + 2y(x) = e^{2x} \,.$$

10. (a) Find the general solution of the following ODE:

$$y''(t) + 4y(t) = 4t^2 \; .$$

(b) For the autonomous differential equation:

$$y'(t) = (y+2)(y-7)$$

determine the equilibrium points and classify them. (Are they stable or unstable?)

Cheat Sheet

Bernoulli Equations: If $\frac{dy}{dx} + p(x)y = q(x)y^n$ is transformed with the change of variables $y = v^{\frac{1}{1-n}}$, then the resulting equation is $\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$ which is linear.

Homogeneous Equations: If $\frac{dy}{dx} = F(y/x)$ is transformed with the change of variables $y = vx$, then the resulting equation is $x\frac{dv}{dx} + v = F(v)$ which is separable.

Variation of parameters: If y_1 and y_2 are linearly independent solutions of $y''(t) + p(t)y'(t) + q(t)y(t) = 0$, then

$$Y(t) := -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\, dt$$

is a solution of $y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$.

Euler Equations: The change of variables given by: $t = \ln x$ transforms the equation:

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = f(x)$$

into the equation:

$$y''(t) + (\alpha - 1)y'(t) + \beta y(t) = f(e^t)\,.$$

Regular Singular Points: If the equation:

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

has a regular singular point at x_0, and we define:

$$\alpha_0 := \lim_{x \to x_0} \frac{(x - x_0) \cdot Q(x)}{P(x)} \quad \text{and} \quad \beta_0 := \lim_{x \to x_0} \frac{(x - x_0)^2 \cdot R(x)}{P(x)},$$

then the indicial equation is given by:

$$r^2 + (\alpha_0 - 1)r + \beta_0 \, .$$

Classifying Linear Systems: If a, b, c, and d are real numbers, and

$$\begin{aligned} x'(t) &= ax(t) + by(t) \\ y'(t) &= cx(t) + dy(t) \, , \end{aligned}$$

then $(0, 0)$ is an equilibrium point.

If we define $D := ad - bc$, and $T := a + d$, then we have the following classifications of the equilibrium point $(0, 0)$:

(a) $D < 0 \Rightarrow$ saddle.

(b) $0 < 4D < T^2$ and

  i. $T < 0 \Rightarrow$ sink.

  ii. $T > 0 \Rightarrow$ source.

(c) $T^2 < 4D$ and

  i. $T < 0 \Rightarrow$ spiral sink.

  ii. $T = 0 \Rightarrow$ center.

  iii. $T > 0 \Rightarrow$ spiral source.

Function	Laplace Transform
$f(t),\ g(t)$	$F(s),\ G(s)$
1	$\frac{1}{s},\quad s > 0$
e^{at}	$\frac{1}{s-a},\quad s > a$
$t^n,\ n \in \mathbb{N},\quad \mathrm{t}^{\mathrm{p}},\ \mathrm{p} > -1$	$\frac{n!}{s^{n+1}},\quad \frac{\Gamma(p+1)}{s^{p+1}},\quad s > 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}},\quad s > a$
$\sin(at),\ \cos(at)$	$\frac{a}{s^2+a^2},\quad \frac{s}{s^2+a^2},\quad s > 0$
$e^{at}\sin(bt),\ e^{at}\cos(bt)$	$\frac{b}{(s-a)^2+b^2},\quad \frac{s-a}{(s-a)^2+b^2},\quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s},\quad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$\delta(t-c)$	e^{-cs}
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$f * g(t) = \int_0^t f(t-T)g(T)\,dT$	$F(s)G(s)$