Algebra Qual August 2015

Attempt all six problems. Start each problem on a new sheet and order them before turning in.

1. Let H and K be subgroups of a group G and let $HK = \{hk \,:\, h \in H, k \in K\}$.
(i) Prove that if $HK = KH$ then HK is a subgroup of G.
(ii) Prove that if $H, K \lhd G$ and $H \cap K = \{1\}$ then $hk = kh \;\; \forall h \in H, k \in K$ and $HK \simeq H \times K$.

2. Prove that a group of order $132 = 2^2 \cdot 3 \cdot 11$ is not simple.

3. Let R be a commutative ring with unity.
(i) Prove that R is a field iff the only ideals of R are $\{0\}$ and R.
(ii) Prove that (x) is a maximal ideal in the polynomial ring $R[x]$ iff R is a field.

4. Let K be the splitting field of $f(x) = x^4 - 3$ over $\mathbb{Q}$.
(i) Find $[K : \mathbb{Q}]$ and $[K : \mathbb{Q}(\sqrt{3})]$.
(ii) Find the group of automorphisms of K that fix $\mathbb{Q}(\sqrt{3})$. Find its proper subgroups and the corresponding fields between $\mathbb{Q}(\sqrt{3})$ and K under the Galois correspondence.

5. Suppose that V and W are finite dimensional F-vector spaces and $T : V \to W$ a linear transformation.
(i) Prove that the kernel $\mathrm{Ker}(T)$ and image $T(V)$ are subspaces of V and W respectively.
(ii) State and prove the relationship (rank-nullity theorem) between the dimensions.

6. Let R be an integral domain and M a unital (unitary) left R-module. For a submodule N of M or ideal I of R define the annihilator

$$\mathrm{Ann}_R(N) = \{a \in R \,:\, an = 0 \;\; \forall n \in N\}, \quad \mathscr{A}nn_M(I) = \{m \in M \,:\, cm = 0 \;\; \forall c \in I\}.$$

i) Prove that $\mathrm{Ann}_R(N)$ is an ideal of R and $\mathscr{A}nn_M(I)$ is a submodule of M.

ii) If M is a free R-module what is $\mathrm{Ann}_R(N)$ and $\mathscr{A}nn_M(I)$?

iii) For the $\mathbb{Z}$-module $M = \mathbb{Z}_{12} \times \mathbb{Z}_{15} \times \mathbb{Z}_{50}$ what is $\mathrm{Ann}_{\mathbb{Z}}(M)$? What is $\mathscr{A}nn_M(3\mathbb{Z})$?
(Here $\mathbb{Z}_n$ denotes the integers mod n and the module action on M is just $r(a,b,c) = (ra, rb, rc)$.)