Algebra Qualifying Exam I
June 2015

1. Let G be a group of order $245 = 5 \cdot 7^2$.
(i) How many Sylow subgroups does G have?
(ii) How many different abelian G are there up to isomorphism?
(iii) Can G be non-abelian? Explain.

2. Prove Cayley's Theorem; namely that any group of order n is isomorphic to a subgroup of the permutation group S_n.

3. Let K be the splitting field of $f(x) = (x^2 - 2)(x^2 + 3)$ over $\mathbb{Q}$ and G its Galois group.
(i) Find $[K : \mathbb{Q}]$ and identify G.
(ii) Find the proper subgroups of G and the corresponding subfields of K under the Galois correspondence.

4. Suppose that U and W are finite dimensional subspaces of an F-vector space V. Prove that

$$\dim_F(U + W) = \dim_F U + \dim_F W - \dim_F(U \cap W).$$

5. Suppose that R is a commutative ring with unity and I an ideal of R.
(i) Prove that R/I is a field iff I is a maximal ideal.
(ii) Is $\mathbb{Z}_5[x]/(x^2 + 1)$ a field?

6. Find the Jordan canonical form for $A = \begin{bmatrix} -1 & 0 & 9 \\ 7 & 6 & -25 \\ 1 & 1 & -2 \end{bmatrix}$ over $\mathbb{C}$. Are there 3×3 matrices with the same eigenvalues as A which are not similar to A over $\mathbb{C}$? Give the Jordan canonical form for each similarity class.