1. (10 pts)

Let $R = k[x, y]$ be the ring of polynomials in two variables over a field k. Consider the ideal $I = (x, y)$. View it as a module over R.

a) Check that the map $F : R^2 \to I$ given by $F : (f, g) \mapsto xf + yg$ is a homomorphism of modules.

b) Check that F is surjective and that $\ker F$ is isomorphic to R as an R-module.

2. **(10 pts)** Let $I \subset \mathbb{C}[x]$ be the ideal generated by $x^3 + x^2 - 2x$. Consider the factor space $V = \mathbb{C}[x]/I$.

a) Find the dimension and a basis of V as a vector space over $\mathbb{C}$.

b) Consider the operator $\varphi : V \to V$ given by multiplication by x. Compute the matrix of φ in the basis constructed in the previous part. What are the rank and nullity of φ?

c) Determine the eigenvalues of the operator φ.

3. (10 pts) Recall, that the center Z of a group G is defined by

$$Z := \{c \in G : \forall g \in G,\ cg = gc\}$$

a) Prove that Z is a subgroup of G.

b) Consider the action of G on itself by conjugation, i.e. an element $g \in G$ acts on an element $h \in G$ by $h \mapsto g^{-1}hg$.

Show that an element of $h \in G$ belongs to Z if and only if the G-orbit of h under the conjugation action consists of one element.

c) Suppose that G is of order p^k, where p is a prime number. Show that the center Z contains more than one element.

Hint: use the Class Equation or the orbits of the conjugation action, and divisibility by p.

4. (10 pts) Let H be a normal subgroup of a group G of index 4. Show that there are either exactly 3 or exactly 5 subgroups of G containing H (including G and H themselves).

5. (10 pts) Let E be the splitting field of $f(x) := x^4 + 7$ over $\mathbb{Q}$.

a) Find all zeros of $f(x)$ in $\mathbb{C}$. (Hint: Use de Moivre's formula.)

b) Prove that $\sqrt[4]{28} \in E$, $\sqrt[4]{28}\, i \in E$, and then that $i \in E$.

c) Show that $\mathbb{Q}(\sqrt[4]{28})$ is a subfield of E of degree 4 over $\mathbb{Q}$.

d) Show that $E = \mathbb{Q}(\sqrt[4]{28}, i)$, $[E : \mathbb{Q}] = 8$, and find a basis for E over $\mathbb{Q}$.

6. (10 pts) Let R be a commutative ring with unity and I, J be ideals in R such that $I + J = R$. (Recall $I + J = \{a + b : a \in I, b \in J\}$.) Prove that

$$R/(I \cap J) \simeq R/I \times R/J.$$