Algebra Qualifying Exam, January 2016

January 30th, 2016

Last name ....

First name ....

KSU Email ....

1. (10 pts)

G is a group of order $351 = 3^3 \cdot 13$.

(i) What are the orders of the Sylow subgroups of G? Must G have a subgroup of order 9?

(ii) Show that G is not simple.

(iii) Suppose that the order of G is 351 and G is cyclic. How many subgroups does G have?

2. (10 pts)

Suppose that G and H are groups and $\phi : G \to H$ is a *surjective* group homomorphism.

(i) Prove that K, the kernel of ϕ, is a normal subgroup of G.

(ii) Prove the first isomorphism theorem, that is $G/K \simeq H$.

(iii) Suppose that B_i is a normal subgroup of A_i for $i = 1, \ldots, n$. Prove that

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \simeq A_1/B_1 \times \cdots \times A_n/B_n.$$

3. (10 pts)

Let K be the splitting field of $f(x) = x^3 - 5$ over $\mathbb{Q}$ and G its Galois group.

(i) Find $[K : \mathbb{Q}]$.

(ii) Describe the elements of G.

(iii) Find the proper subgroups of G and the corresponding subfields of K under the Galois correspondence.

4. (10 pts)

Suppose that V is an n-dimensional vector space V over a field F, and W is an m-dimensional subspace of V.

(i) Prove that a basis for W can be extended to a basis for V.

(ii) Prove that V/W is an F-vector space of dimension $n - m$.

5. (10 pts)

Suppose that R is a commutative ring with unity.

(i) Prove that the ideal in $R[x]$ generated by x is a prime ideal iff R is an integral domain.

(ii) Prove that $\mathbb{Q}[x]$ is a principal ideal domain. Give a generator for the ideal generated by two polynomials $x^2 - 4x + 3$ and $x^2 - 9$.

(iii) Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

6. (10 pts) Let A be a linear transformation of a complex finite-dimensional vector space satisfying the property $A^2 = A$.

(i) Prove that $\operatorname{Im} A \cap \operatorname{Ker} A = \{0\}$.

(ii) Prove that $V = \operatorname{Im} A \oplus \operatorname{Ker} A$.

(iii) Describe how the Jordan canonical form for A looks like.