



Algebra Qualifying Exam, August 2017

August 25th, 2017

Last name

First name

KSU Email



543064D9-D52A-4D22-B5BA-27D09FF5D23E

algebra-qe-i-august-2017-a00af

#1 2 of 18

**1. (10 pts)**

For G a group, we say that $g \in G$ is an involution if $g^2 = e$ where e is the identity element. Suppose G is a finite group such that all elements of G are involutions.

- a) Prove that $|G| = 2^k$ for some integer $k \geq 0$.
- b) Prove that G is commutative.



0EB761A0-1DED-4294-9A2B-E6190904A875

algebra-qe-i-august-2017-a00af

#1 4 of 18



2. (10 pts) Let I and J be ideals of R . We write

$$I + J = \{i + j : i \in I, j \in J\},$$

and

$$I * J = \{ij : i \in I, j \in J\}.$$

- a) Is $I + J$ necessarily an ideal of R ? Prove or provide a counterexample.
- b) Is $I * J$ necessarily an ideal of R ? Prove or provide a counterexample.



C9425700-95FE-4841-BD5B-732AB1E6BF9A

algebra-qe-i-august-2017-a00af

#1 6 of 18



3. (10 pts) Let $A : V \rightarrow V$ be a linear transformation of a finite-dimensional vector space V satisfying the property $A^2 = A$.
- a) Prove that $\text{Im } A \cap \ker A = \{0\}$.
 - b) Prove that $V = \text{Im } A \oplus \ker A$.
 - c) Suppose $\dim V = n$ and $\text{rank } A = k$. What is the Jordan form of A ?



CC75FFAD-239D-4D26-81FC-9B13FB66B83C

algebra-qe-i-august-2017-a00af

#1 8 of 18



4. (10 pts) Provide an example of three fields $F \subset E \subset L$ such that E/F and L/E are Galois, but L/F is not.

Hint: One can use subfields of the splitting field of $x^4 + 2$ for this problem.



1DBDC107-6A78-41A7-B246-9D13FE4220AC

algebra-qe-i-august-2017-a00af

#1 10 of 18



5. (10 pts) Let R be a ring with identity and let M be a left R -module. Recall that the *annihilator* of M in R is

$$\text{ann}_R(M) := \{r \in R : \forall m \in M, rm = 0\}.$$

- a) Prove that $\text{ann}_R(M)$ is a two-sided ideal of R .
- b) Note that an abelian group is a $\mathbb{Z}$ -module. How many possibilities, up to isomorphism, are there for an abelian group M of order 400 with $\text{ann}_{\mathbb{Z}}(M)$ the ideal generated by $20 \in \mathbb{Z}$?



F2845650-6C4E-48FE-B77F-DC5DCF30B152

algebra-qe-i-august-2017-a00af

#1 12 of 18



6. (10 pts)

- a) How many distinct actions of the group $\mathbb{Z}$ are there on the set $\{1, 2, 3, 4\}$?
- b) How many distinct **transitive** group actions of $\mathbb{Z}$ are there on the set $\{1, 2, 3, 4\}$? (Recall that an action of a group G on a set X is *transitive* if for every $x \in X$ we have $\{gx : g \in G\} = X$.)



BEBE0BB7-830D-4B82-A198-981174FFB3EF

algebra-qe-i-august-2017-a00af

#1 14 of 18

29F0EE65-7C02-4B3E-AF21-28F5DF112E50

algebra-qe-i-august-2017-a00af

#1 15 of 18





355B783E-60C1-4CA6-BAF3-8D06E56A3CD6

algebra-qe-i-august-2017-a00af

#1 16 of 18

00A433A9-BEB2-4938-9672-2C4D97A8D0F4

algebra-qe-i-august-2017-a00af

#1 17 of 18





0AB00042-5006-4031-95FC-8428E1B9B5EA

algebra-qe-i-august-2017-a00af

#1 18 of 18