Algebra Qualifying Exam, August 2017

August 25th, 2017

Last name ....

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1. (10 pts)

For G a group, we say that $g \in G$ is an involution if $g^2 = e$ where e is the identity element. Suppose G is a finite group such that all elements of G are involutions.

a) Prove that $|G| = 2^k$ for some integer $k \geq 0$.

b) Prove that G is commutative.

2. (10 pts) Let I and J be ideals of R. We write

$$I + J = \{i + j : i \in I, j \in J\},$$

and

$$I * J = \{ij : i \in I, j \in J\}.$$

a) Is $I + J$ necessarily an ideal of R? Prove or provide a counterexample.

b) Is $I * J$ necessarily an ideal of R? Prove or provide a counterexample.

3. (10 pts) Let $A : V \to V$ be a linear transformation of a finite-dimensional vector space V satisfying the property $A^2 = A$.

a) Prove that $\operatorname{Im} A \cap \ker A = \{0\}$.

b) Prove that $V = \operatorname{Im} A \oplus \ker A$.

c) Suppose $\dim V = n$ and $\operatorname{rank} A = k$. What is the Jordan form of A?

4. (10 pts) Provide an example of three fields $F \subset E \subset L$ such that E/F and L/E are Galois, but L/F is not.

Hint: One can use subfields of the splitting field of $x^4 + 2$ for this problem.

5. (10 pts) Let R be a ring with identity and let M be a left R-module. Recall that the *annihilator* of M in R is

$$\operatorname{ann}_R(M) := \{r \in R : \forall m \in M,\ rm = 0\}.$$

a) Prove that $\operatorname{ann}_R(M)$ is a two-sided ideal of R.

b) Note that an abelian group is a $\mathbb{Z}$-module. How many possibilities, up to isomorphism, are there for an abelian group M of order 400 with $\operatorname{ann}_{\mathbb{Z}}(M)$ the ideal generated by $20 \in \mathbb{Z}$?

6. (10 pts)

a) How many distinct actions of the group $\mathbb{Z}$ are there on the set $\{1, 2, 3, 4\}$?

b) How many distinct **transitive** group actions of $\mathbb{Z}$ are there on the set $\{1, 2, 3, 4\}$? (Recall than an action of a group G on a set X is *transitive* if for every $x \in X$ we have $\{gx : g \in G\} = X$.)