1. (10 pts) Let F be a field, and consider the ring $F[x]$ of polynomials in one variable with coefficients in F.

a) Show that $F[x]$ is a vector space of infinite dimension over F.

b) Construct a linear transformation $\phi : F[x] \to F[x]$ which is injective, but not surjective.

c) Construct a linear transformation $\psi : F[x] \to F[x]$ which is surjective, but not injective.

2. (10 pts)

a) Let K/F be a field extension, and M and N be square matrices over F. Show that M and N are similar over F if and only if they are similar over K.

Hint: What do you know about the Rational Canonical Form?

b) Let M be a square matrix over $\mathbb{R}$. Show that M is similar to its transpose.

Hint: Use Part (a), with $K = \mathbb{C}$ and the Jordan Canonical Form.

3. (10 pts) Let p be a positive prime integer. Consider the set

$$G := \{\theta \in \mathbb{C} : \exists n \in \mathbb{Z}_{\geq 0},\ \theta^{p^n} = 1\}$$

a) Show that G is an infinite group under multiplication.

b) Show that every proper subgroup of G is finite.

4. (10 pts) Let $N \triangleleft G$ be a normal subgroup of a group G, and let $P < N$ be a Sylow subgroup of N for some prime number p. Show that $G = N_G(P)N$, i.e. that any element $g \in G$ can be written as a product $g = hn$, where $n \in N$ and h is such that $h^{-1}Ph = P$.

Hint: Given a $g \in G$, what can one say about the conjugate subgroup $g^{-1}P$g? Where does it lie?

5. (10 pts)

a) Show that the polynomials x^4+2 and x^4-8 have the same splitting field $K \subset \mathbb{C}$ over $\mathbb{Q}$.

b) Find $[K : \mathbb{Q}]$ and compute the Galois group G of the extension.
Hint: Use that G is a subgroup of S_4.

c) How many subfields $E \subset K$ such that $[K : E] = 2$ exist? Identify all of them.

6. (10 pts) Let R be a commutative ring with 1. Denote

$$I := \{r \in R : \exists n \in \mathbb{Z}_{>0},\ r^n = 0\}.$$

a) Show that I is an ideal.

b) Show that if I is maximal, then for every $x \in R$ either $x \in I$ or x is a unit.