Algebra QE I Exam
Spring 2018

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all six problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

Let a group G with 35 elements act on a set X with 18 elements. Prove that there is an element $x \in X$ whose stabilizer is G.

2. (10 pts)

a) Let G be an abelian group. Show that the following are equivalent:

   i. G is not cyclic.

   ii. There is a prime number q such that there are more than $q-1$ elements of order q in G.

b) Let p be prime. Show that the multiplicative group of units in $\mathbb{Z}_p$ is cyclic.

c) Is it true that the multiplicative group of units in $\mathbb{Z}_n$ is cyclic for any n? Prove it or provide a counterexample.

3. (10 pts)

a) Is $\mathbb{Z}[x, y]$ a Euclidean domain? Prove your answer.

b) Is $\mathbb{Z}[x, y]$ a principal ideal domain? Prove your answer.

c) Is $\mathbb{Z}[x, y]$ a unique factorization domain? You don't have to prove your answer.

4. (10 pts)

Consider the ring $R = \mathbb{C}[x]$. Denote by $M = \mathbb{C}[x, y]/(x^2+y^2)$. Then M is a module over R with the module structure given by multiplication modulo $x^2 + y^2$. Show that M is finitely generated over R and find a finite set of generators.

5. (10 pts)

Let A be an $n \times n$ matrix of a complex linear transformation such that $A^2 = A$. Prove that A is diagonalizable.

6. (10 pts) Let p, q be prime numbers. Let E be the field with p^q elements and let $F \subset E$ be the field with p elements.

a) Find the index $[E : F]$.

b) Let α be any element of E that does not lie in F. Prove that the degree of the minimal polynomial of α is q.

c) Prove that E/F is a Galois extension and find its Galois group.