Algebra QE I Exam
August 2019

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

(a) Let p be a prime number. Show that the dihedral group D_{2p} with $2p$ elements has exactly one non-trivial normal subgroup.

(b) Show that if n is not prime, then D_{2n} has more than one non-trivial normal subgroup.

2. (10 pts) Suppose G is a group of order 45 containing an element of order 9. Prove that G is a cyclic group.

3. (10 pts)

Let R be commutative ring with identity. Suppose $P, I_1, I_2, \ldots, I_n$ are ideals in R with P a prime ideal. Prove that if $I_1 I_2 \cdots I_n \subseteq P$ then $I_j \subseteq P$ for some $1 \leq j \leq n$.

4. (10 pts) Let V be an n-dimensional vector space over $\mathbb{C}$ with basis $\{e_1, \ldots, e_n\}$ and $S : V \to V$ a linear operator defined by the property $S(e_i) = e_{i+1}$ for $1 \le i \le n-1$ and $S(e_n) = e_1$.

(a) What are the characteristic and minimal polynomials of S?

(b) What is the Jordan canonical form of S?

5. (10 pts)

Let R be a PID, and M a finitely generated R-module. Recall that the torsion submodule of M is

$$\mathrm{Tor}(M) = \{m \in M : \text{there is a non-zero } r \in R \text{ such that } rm = 0\}$$

(a) Show that if $\mathrm{Hom}(M, R) = 0$, then M is a torsion module (i.e., $M = \mathrm{Tor}(M)$).

(b) Prove that there is an R-module isomorphism

$$\mathrm{Hom}(M, R) \cong \frac{M}{\mathrm{Tor}(M)}$$

6. (10 pts) Let F be a field of characteristic zero, and $f(x) \in F[x]$ an irreducible polynomial of degree 4.

(a) What are the possible degrees of the splitting field of $f(x)$? Explain your response.

(b) Let K be a splitting field of $f(x)$ such that K is a quadratic extension of the intermediate field E which is not Galois over F. Suppose also that every non-trivial intermediate field has even degree over F. What is the Galois group of K over F?