Algebra QE I Exam
June 2019

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

Suppose G is a finite abelian group and there is no non-trivial isomorphism $\psi : G \to G$ which has order 2. Prove that G is either the trivial group $\{1\}$ or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

2. (10 pts)

Suppose V is an n-dimensional vector space over F with $n \geq 1$ and $T : V \to V$ is a linear operator with $\ker(T) = \operatorname{im}(T)$.

(a) Show $\dim V$ is even.

(b) Find the characteristic polynomial of T.

(c) Find the minimal polynomial of T.

(d) What is the Jordan normal form of T? Explain your response.

3. (10 pts) Let R be a PID with I and J ideals in R. Prove that $IJ = I \cap J$ if and only if $I + J = R$.

4. (10 pts)

Let R be a commutative ring with unit and M an R-module.

(a) For $m \in M$, define the annihilator $\mathrm{Ann}_R(m)$ of m as the subset

$$\{r \in R : rm = 0\}.$$

Prove that $\mathrm{Ann}_R(m)$ is an ideal in R.

(b) Suppose M is generated by m as an R-module. If $\mathrm{Ann}_R(m)$ is a maximal ideal show that M has no non-trivial submodules.

5. (10 pts)

Let F be a field, $f(x)$ the polynomial $x^4+1 \in F[x]$ and K its splitting field.

(a) Prove that $f(x)$ is separable if and only if the characteristic of F does not equal 2.

(b) Prove that $[K : F] \leq 4$.

6. (10 pts) Prove that $\mathbb{Q}(i, \sqrt{2})$ over $\mathbb{Q}$ is a Galois extension and find its Galois group.