Algebra QE I Exam, August 2020

Problem 1. Let $J = J_n(0)$ be the Jordan block matrix $n \times n$ with eigenvalue 0. Find the Jordan canonical form of J^2.

(**Hint:** consider odd and even cases separately.)

Problem 2. Describe all groups of order 9 up to isomorphism. For one of these groups exhibit two non-commuting automorphisms.

Problem 3. Let G be a group. By a maximal proper subgroup of G we mean a subgroup $M \subset G$ such that $M \neq G$ and the only subgroups of G containing M are M and G.

(1) Describe all the maximal subgroups of the dihedral group of order 10.
(2) Show that if a maximal subgroup $M \subset G$ is normal, then the index of M in G is finite and prime.

Problem 4. Let K be the splitting field of the polynomial $x^4 - x^2 - 1$ over $\mathbb{Q}$. Compute the Galois group of the extension $K/\mathbb{Q}$.

Problem 5. Recall that an R-module N is called *Noetherian* if for any growing tower of submodules

$$L_1 \subset L_2 \subset \ldots \subset L_k \subset \ldots \subset M$$

there exists a positive integer n such that $L_n = L_{n+1} = \ldots$.

Let R be a commutative ring with identity, M be an R-module, and $f : M \to M$ be a surjective R-module homomorphism.

(1) Prove that if in addition M is Noetherian then f is an isomorphism.
(2) Provide an example of R, M and f, where M is not Noetherian over R and f is not an isomorphism.

Problem 6. Describe all zero-divisors and all units in the quotient rings

(1) $R := \mathbb{Q}[x]/(x^2 - 1)$,
(2) $S := \mathbb{Q}[x]/(x^2 + 1)$.