Algebra QE I Exam, June 2020

Instructions: This is a three-hour, closed-book exam: no resources, including texts, notes, the internet, or communication with other people, are to be used. Solving four problems fully and correctly will earn a pass; in some cases solved parts of problems may be combined to count for a full problem.

Problem 1. Prove or disprove the following: a group is abelian if and only if every one of its subgroups is normal.

Problem 2. Describe all groups of order 55 up to isomorphism.

Problem 3. Let $R = \mathbb{Z}[\sqrt{-6}]$.

(1) Show that 2 and $\sqrt{-6}$ are irreducibles in R. Hint: use the norm.
(2) Show that R is not a Unique Factorization Domain (hence not a P.I.D.)
(3) Give an explicit ideal in R which is not principal.

Problem 4. Let R be a Principal Ideal Domain, $p \in R$ be a prime, and $a \in R \setminus \{0\}$ be a non-zero element. Let n be the maximal integer such that a is divisible by p^n. Consider the R-modules $M := R/(a)$ and

$$p^k M := \{c \in M \mid \exists b \in M,\, c = p^k b\}$$

for non-negative integers k. Prove that

$$p^k M = \begin{cases} (p^k)/(a) & \text{if } k < n, \\ (p^n)/(a) & \text{if } k \geq n. \end{cases}$$

Problem 5. Find the Jordan Canonical Form of the matrix

$$M = \begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Does there exist a matrix L with the same characteristic and minimal polynomials as M, but not similar to M?

Problem 6. Let n be a positive integer, $p_n := x^4 + n$, and E_n be the splitting field of p_n over $\mathbb{Q}$.

(1) Find all positive integers n such that p_n is irreducible over $\mathbb{Q}$.
(2) Show that E_n coincides with the splitting field of $x^4 - 4n$ for all positive integers n.
(3) Find all positive integers n such that $[E_n : \mathbb{Q}] = 4$.