ALGEBRA QE I AUGUST 2021

(1) Prove that S_4 is generated by (1234) and (1243).

(2) Let G be a group of order 30, and let P and Q be its Sylow subgroups of orders 5 and 3 respectively. Show that either P or Q is normal. Deduce that G contains a cyclic subgroup of order 15.

(3) Let R be a commutative ring with identity $1 \neq 0$. Show that an ideal M in R is maximal if and only if the quotient ring R/M is a field.

(4) Let $M \subset \mathbb{C}[x, y]$ be the module over the ring $\mathbb{C}[x, y]$ consisting of all polynomials that are sums of monomials of degree at least 3. For example, the polynomial $xy^2 + 2x^2y^3 - y^{10}$ is in M, but the polynomial $x + 2x^2y^3 - y^{10}$ is not in M since the degree of the monomial x is 1. Find a finite set of generators for M.

(5) Let F be the splitting field of $x^5 + 1$ over $\mathbb{Q}$. Show that F is a Galois extension of $\mathbb{Q}$ and find $\mathrm{Gal}(F/\mathbb{Q})$.

(6) Let $U \subset V \subset W$ be vector spaces over a field F with $\dim U = 1$, $\dim V = 2$, $\dim W = 3$. Let $T : W \to W$ be a linear transformation such that $T(W) \subset U$ and $T(V) = 0$. Find the Jordan Normal Form of T assuming $T \neq 0$.