ALGEBRA QE I JUNE 2021

(1) Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

(2) Show that if H and K are subgroups of a group, then HK is a subgroup if and only if $HK = KH$.

(3) Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$. Recall that two elements a and b of a commutative ring are associate if $a = bc$ where c is an invertible element. Prove that $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ are irreducibles in R, no two of which are associate in R, and that $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ are two distinct factorizations of 6 into irreducibles in R.

(4) Let R be the subset of the ring of 2×2 matrices with integral coefficients that consists of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with even b, c. Let $M = \mathbb{Z} \times \mathbb{Z}$ viewed as a module over R with the standard linear action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Is M a cyclic module? If yes, provide a generator, if no, prove it is not.

(5) Let $E \subset F$ be a Galois extension with Galois group $S_3 \times \mathbb{Z}_3$.
  (a) Find the number of distinct fields K with $E \subset K \subset F$ such that $E \subset K$ is an extension of degree 3.
  (b) Find the number of distinct fields K with $E \subset K \subset F$ such that $E \subset K$ is a Galois extension of degree 3.

(6) Let A be a complex $n \times n$ matrix such that $A^2 = \mathrm{Id}$.
  (a) Prove that A is diagonalizable.
  (b) Describe all possibilities for the characteristic polynomial of A.