ALGEBRA QUAL DRAFT

AUGUST 2022

All rings are assumed to be unitary (i.e., with 1) and all ring homomorphisms are assumed to send 1 to 1. A ring is assumed to be possibly noncommutative unless stated explicitly.

(1) Let G be a group of order 12 and let H be a group of order 15. Prove that any subgroup of $G \times H$ of order 9 is commutative.

(2) Let G be a finite group acting on a finite set X. Let X/G be the set of all G-orbits in X. For any $x \in X$, the stablizer subgroup of x is defined to be $G_x = \{g \in G \mid gx = x\}$. It is known that, if $y = hx$ for some $h \in G$ then $G_y = hG_x h{-}1$. For each orbit $O \in X/G$, the number $|G_x|$ is independent of the choice of $x \in O$ and is denoted by $|\operatorname{Aut}(O)|$. Prove the following identity

$$\frac{|X|}{|G|} = \sum_{O \in X/G} \frac{1}{|\operatorname{Aut}(O)|}.$$

(3) Let R be a commutative ring. For any field F and any ring homomorphism $\phi : R \to F$, it is known that the kernel $\ker(\phi) \subseteq R$ is an ideal of R. Prove that an ideal I of R is prime if and only if $I = \ker(\phi)$ for some ring homomorphism $\phi : R \to F$ with F being a field F.

(4) Find the number of elements in the quotient ring $\mathbb{Z}[x]/\langle 3, x^4 + 2x + 2\rangle$. Here $\mathbb{Z}$ is the ring of integers and $\langle a, b\rangle$ denotes the two sided ideal generated by a, b in a ring.

(5) Let F be a field (not necessarily algebraically closed). Let A be an $n \times n$ matrix with entries in F such that the characteristic polynomial of A is $\det(xI - A) = (x - \lambda)^n$ for some $\lambda \in F$. Prove that there is an invertible $n \times n$ matrix P with entries in F such that PAP^{-1} is an upper triangular matrix over F.

(6) Let $E \subset F \subset G$ be field extensions such that all of $E \subset F$, $F \subset G$, and $E \subset G$ are Galois extensions. Suppose that the Galois group of the extension $E \subset F$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and the Galois group of the extension $F \subset G$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Find the Galois group of the extension $E \subset G$. You have to prove that your answer is the only possible one.