Algebra QE I June 2022

INSTRUCTIONS

This is a closed-book exam. No written material, electronic tools, or communication with others is permitted.Four questions correctly solved (up to minor errors) will earn a pass on this exam. Parts of questions may in some cases combine to count as one full problem.

Do not write your name on any page of your work.

You have three hours to do your work and fifteen minutes to upload the solutions.

Submitting your work:

- Find the email from Crowdmark Mailer with subject "Graduate Program 2021–2022 – New Assignment: Algebra" in the subject line. Follow steps given.
- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to sarahrez@ksu.edu.
- Before leaving the library, give all hard copies to the proctor.

All rings are assumed to be unitary (i.e., with 1) and all ring homomorphisms are assumed to send 1 to 1. A ring is assumed to be possibly noncommutative unless stated explicitly.

(1) Let H be a subgroup of a group G. Consider the subgroup

$$L = \{(h,h) \mid h \in H\}$$

of $H \times G$. Prove that L is a normal subgroup in $H \times G$ if and only if H is contained in the center of G.

(2) Let $GL_n(\mathbb{F}_q)$ be the group of all invertible $n \times n$ matrices over $\mathbb{F}_q$, a finite field with $q = p^r$ elements, where p is a prime and $r \geq 1$ is an integer. Let $B_n(\mathbb{F}_q)$ be the subgroup of $GL_n(\mathbb{F}_q)$ consisting of all invertible upper triangular matrices and $U_n(\mathbb{F}_q)$ the subgroup of $GL_n(\mathbb{F}_q)$ consisting of all upper triangular matrices with all diagonal entries being 1. You are given the following facts:

(i) $GL_n(\mathbb{F}_q)$ is a group, and $B_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_q)$ are subgroups of $GL_n(\mathbb{F}_q)$;

(ii) The order of the group $GL_n(\mathbb{F}_q)$ is $|GL_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$.

(iii) The group $B_n(\mathbb{F}_q)$ is the normalizer of $U_n(\mathbb{F}_q)$ in $GL_n(\mathbb{F}_q)$.

Do the following:

(a) Show that $U_n(\mathbb{F}_q)$ is a Sylow p-subgroup of $GL_n(\mathbb{F}_q)$.

(b) Show that the number of Sylow p-subgroups is $|GL_n(\mathbb{F}_q)/B_n(q)| = \prod_{r=1}^{n} \frac{q^r-1}{q-1}$.

(3) Find all prime ideals in the ring $R = \mathbb{C}[x,y]/\langle x^2 + y^3 - 2\rangle$ that contain $\bar{x}$. Here $\bar{x}$ is the image of x in R. For each ideal, make sure to prove it is prime. (*Hint: Consider the quotient ring* $R/\langle \bar{x}\rangle$.)

(4) Let R be a ring and M be a left R-module. For each element $m \in M$, define

$$\mathrm{Ann}_R(m) = \{r \in R \mid rm = 0\}.$$

Prove that $\mathrm{Ann}_R(m)$ is a left ideal of R. Give an example of R, M, and $m \in M$ such that $\mathrm{Ann}_R(m)$ is not a right ideal.

(5) A square matrix A over a field F is called nilpotent if $A^k = \overbrace{AA\cdots A}^{k \text{ times}} = 0$ for some positive integer k.

(a) Prove that if A is an $n \times n$ nilpotent matrix over F and B is an $n \times n$ invertible matrix over F, then BAB^{-1} is nilpotent.

(b) Let $\mathcal{N}_n(F)$ be the set of all $n \times n$ nilpotent matrices over an algebraically closed field F. Part (a) shows that the group $GL_n(F)$ of all invertible $n \times n$ matrices over F acts on the set $\mathcal{N}_n(F)$ by $B \cdot A = BAB^{-1}$ for all $A \in \mathcal{N}_n$ and $B \in GL_n(F)$. Find the number of $GL_n(F)$-orbits in $\mathcal{N}_n(F)$ for $n = 3$.

(6) Let $F \subseteq E = F(\alpha)$ be a simple algebraic field extension. For any field extension $F \subseteq K$, define the set

$$\mathrm{Hom}_F(E, K) = \{\sigma : E \to K \mid \sigma \text{ is a ring homomorphism with } (\forall a \in F)\ \sigma(a) = a\}.$$

Prove $|\,\mathrm{Hom}_F(E, K)| \leq [E : F]$.