# Algebra Qualifying Exam
## June 2023

*All rings are assumed to be unitary (i.e., with 1) and all ring homomorphisms are assumed to send 1 to 1. A ring is assumed to be possibly noncommutative unless stated explicitly.*

1 Let $G$ be a group and $H$ be a subgroup of $G$.
  (a) Show that the set of left cosets of $H$ in $G$ form a partition of $G$.
  (b) Assume that $G$ is finite. Prove Lagrange's theorem, i.e., any subgroup $H$ of $G$ has order $|H|$ being a factor of $|G|$.

2 Let $G$ be a group of order 99 containing an element of order 9. Prove that $G$ is cyclic.

3 Let $R$ be a commutative ring and $I$ be an ideal in $R$. Define

$$\text{rad } I = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$$

  (a) Show that rad $I$ is an ideal in $R$ containing $I$.
  (b) Let $R = \mathbb{Z}$ and $I = (k)$ the ideal generated by the integer $k$. Show that $\text{rad}(k) = (k)$ if and only if $k$ is a product of distinct primes.

4 Let $S$ be a ring and $R \subseteq S$ be a subring. For any left $R$-module $M$, regarding $S$ as a left $R$-module, define a map $\pi : S \times \text{Hom}_R(S, M) \to \text{Hom}_R(S, M)$ by $\pi(s, f) = (sf)$ with $(sf)(x) = f(xs)$ for all $x \in S$. Show that $\pi$ defines a left $S$-module structure on $\text{Hom}_R(S, M)$.

5 Let $\mathsf{k}$ be a field and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a monic polynomial in $\mathsf{k}[x]$. Let $V = \mathsf{k}[x]/\langle f(x)\rangle$ be the vector space over $\mathsf{k}$. The matrix $A$ of the linear transformation $\phi : V \to V$ defined by multiplication of $x$ under the standard basis $\{1 = x^0, x^1, x^2, \cdots, x^{n-1}\}$ is called the companion matrix of $f(x)$. Write the companion matrix $A$ and prove $\det(xI_n - A) = f(x)$.

6 Let $F \subseteq K$ be a field extension. Show that an element $\alpha \in K$ is algebraic if and only if the image of the map $ev_\alpha : F[x] \to K$, defined by $ev_\alpha(f(x)) = f(\alpha)$ is a field.