# Algebra QE 1

## May 29, 2024

Name: ______________________________

KSU Email: ______________________________

**Instructions:** You have three hours to do your work and fifteen minutes to upload the solutions.

0) Four full problems correctly done will earn a pass. Parts of several problems may combine to count for a full problem.

1) Do not write your name on any of the pages.

2) This is a closed book exam: no books, no notes, no calculators etc. Only plain papers and pens should be on your table.

3) You must have your camera on showing your hands and, if possible, at least part of your faces during the exam.

4) After you receive the exam online by email, if you want you can print it or you can keep it open on your laptop or cellphone. After that you are allowed to use your computer or any electronic device during the exam only to read the exam and later to upload it on Crowdmark. Also you can communicate to the examiner via private chat in Zoom.

5) In case you lose internet connection at some point, you can continue your exam, however the examiners might consider to have an oral reexamination with you where you would need to explain steps in your work. You can be asked additional questions. In case the loss of connection is long, you can contact the examiner.

6) The exam is supposed to take 3 hours not counting the time of printing or accessing the problems and uploading your test. If you want you can have a bit of extra time, but the exam must be uploaded to Crowdmark no less than 10 minutes after the exam

Submitting your work:

- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to tinaande@ksu.edu.

**Problem 1**. $2024 = 2^3 \cdot 11 \cdot 23$.

(a) Show that a group of order 2024 cannot be simple.

(b) List all abelian groups of order 2024 up to isomorphism.

**Problem 2**. Let $R$ be a *finite* commutative ring with identity. Prove that every prime ideal in $R$ is maximal.

**Problem 3**. Let $I$ be a proper ideal in an *integral domain* $R$. Let $p(x)$ be a non-constant monic polynomial in $R[x]$.

(a) Prove that if the image of $p(x)$ in $(R/I)[x]$ cannot be factored into polynomials of smaller degree then $p(x)$ is irreducible in $R[x]$.

(b) Show that the polynomial $p(x) = x$ is irreducible in $\mathbb{Z}[x]$, $(\mathbb{Z}/5\mathbb{Z})[x]$, $(\mathbb{Z}/2\mathbb{Z})[x]$ but is *not* irreducible $(\mathbb{Z}/10\mathbb{Z})[x]$. Find all factorizations of $p(x)$ in $(\mathbb{Z}/10\mathbb{Z})[x]$.

**Problem 4.** Let $R$ be a commutative ring with 1. An $R$-module $M$ is called *Noetherian* if every ascending chain of submodules of $M$ becomes stationary; that is, for any chain

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

there exists an index $n$ such that $N_n = N_{n+1} = N_{n+2} = \cdots$.

Prove that an $R$-module $M$ is Noetherian *if and only if* every submodule of $M$ is finitely generated.

**Problem 5**. Determine all similarity classes of rational $3 \times 3$ matrices $A$ satisfying the equation $A^4 = A^2$. Provide a representative for each class. (Two matrices $A$ and $B$ are called similar if there exists an invertible matrix $P$ such that $B = P^{-1}AP$.)

**Problem 6**. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 3, and $G$ its Galois group. Prove that if $G$ is the cyclic group of order 3, then $f(x)$ splits completely over $\mathbb{R}$.