Algebra QE I Sample Exam

Attempt all problems. Justify your answers. Please use a separate sheet of paper for each problem.

1. Recall that the *centralizer* of a subgroup H of a group G is the set

$$C_c(H) = \{g \in G \,|\, \forall h \in H \; ghg^{-1} = h\}.$$

   (a) Prove that if H is normal in G then $C_c(H)$ is normal in G.

   (b) Assume H is normal. Prove that $G/C_c(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$, the group of automorphisms of H.

2. Let T be an invertible linear operator on a finite-dimensional vector space. Prove that T^{-1} is diagonalizable if and only if T is diagonalizable.

3. Let $M_{n\times n}(\mathbb{F})$ denote the ring of $n \times n$ matrices with entries in a field $\mathbb{F}$. Prove that this ring has no two-sided ideals except $M_{n\times n}(\mathbb{F})$ and $\{0\}$.

4. Find the splitting field and Galois group of $f(x) = (x^4 - 4)$ and identify the intermediate fields corresponding to its subgroups.

5. Prove that if G is a group of order p^n, where p is a prime number, then the center of G, $Z(G)$, cannot equal $\{e\}$. Hint: use the class equation.

6. Let I be a non-zero ideal of a commutative ring R with identity. Prove that I is a free R-module if and only if $I = Ra$ for some $a \in R$ that is not a zero divisor.