Analysis QE I Exam
August 2015

Attempt at most six problems, chosen from any of the sections (which correspond, roughly, to material from 721, 722, and 700-level complex analysis). Justify your answers. Please use a separate sheet of paper for each problem.

Section I

1. Assume (a_n) is a convergent sequence in a metric space (X, d). Show that there is a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that the series $\sum_{k=1}^{\infty} d(a_{n_k}, a_{n_{k+1}})$ converges.

2. Using the definition of uniform continuity, show that any uniformly continuous function $f : (0,1) \to \mathbb{R}$ is bounded.

3. Let $r_1, r_2, \ldots r_n$ be real numbers in $[0,1]$ where $n \in \mathbb{N}^+$. Let $f : [0,1] \to \mathbb{R}$ be the characteristic function of $\{r_1, r_2, \ldots r_n\}$; i.e,

$$f(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Using the definition of the Riemann integral, prove that f is Riemann integrable.

Section II

4. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 f(x) x^n \, \mathrm{d}x = 0 \quad (n = 0, 1, 2, \ldots).$$

Prove that f is identically zero on $[0,1]$. Hint: prove that $\int_0^1 f^2(x) \, \mathrm{d}x = 0$.

5. Consider the set of points (x, y) in the real plane that satisfy $x + \sin(xy) = 0$.
   (a) Is there a neighborhood of the origin on which this set is the graph of a function $y = f(x)$?
   (b) Is there a neighborhood of the origin on which this set is the graph of a function $x = f(y)$?

6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Show that f is continuous at $(0,0)$.
   (b) Show that the directional derivatives $D_u f$ exist at $(0,0)$ (u a unit vector) and compute them.
   (c) Show that f is not differentiable at $(0,0)$.

Section III

7. Give two Laurent Series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

   and specify the regions in which those expansions are valid.

8. Use residues to compute the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2+\cos\theta}.$$

9. Find all points z where the function $f(z) = \mathrm{Re}(z) \cdot \mathrm{Im}(z)$ is complex differentiable.