Analysis QE I Exam

Attempt at most six problems, chosen from any of the sections (which correspond respectively to material from 721, 722, and 700-level complex analysis). Justify your answers. Please use a separate sheet of paper for each problem.

Section I

1. Let (X, d) be a metric space. Prove that if a Cauchy sequence in X has a convergent subsequence then the sequence converges.

2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and is periodic with period 1, i.e.,

$$f(x+1) = f(x) \text{ for all } x \in \mathbb{R}.$$

Prove that f is uniformly continuous.

3. Use the definition of the Riemann integral to prove that if $a < b < c$ are real numbers and f is Riemann integrable on both $[a, b]$ and on $[b, c]$, then f is Riemann integrable on $[a, c]$.

Section II

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0). \end{cases}$$

   (a) Show that f is continuous at $(0, 0)$.

   (b) Show that the directional derivatives $D_u f$ exist at $(0, 0)$, and compute them.

   (c) Show that f is not differentiable at $(0, 0)$.

5. Can the equation $(x^2 + y^2 + 2z^2)^{1/2} = \cos z$ be solved uniquely for y from x and z in a neighborhood of $(0, 1, 0)$? For z in terms of x and y?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that there is a sequence p_n of polynomials such that for every $R > 0$, the sequence converges uniformly to f on the interval $[-R, R]$.

Section III

7. Find the Laurent series for $f(z) = \dfrac{z^2+1}{z(z-3)}$ in the annulus $0 < |z| < 3$.

8. Let G be a connected open subset of $\mathbb{C}$ and f and g analytic functions in G such that $f(z)g(z) = 0$ for all $z \in G$. Prove that either $f \equiv 0$ or $g \equiv 0$.

9. Let u and v be real harmonic functions and suppose that v is the harmonic conjugate of u. Show that

$$\frac{u}{u^2+v^2} \quad \text{and} \quad \frac{-v}{u^2+v^2}$$

are both harmonic, assuming $u^2+v^2 \neq 0$.