1. (10 pts) Let $K \subset \mathbb{R}$ be a set with the following property: every continuous function $f : K \to \mathbb{R}$ is bounded. Prove that K is closed and bounded (hence compact).

2. (10 pts) Let a_n be a sequence of positive real numbers, such that

$$\sum_{n=1}^{\infty} a_n$$

diverges. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$$

also diverges.

3. (10 pts)

Recall that the Dirichlet function $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

Show that the Dirichlet function is not Riemann integrable.

4. (10 pts) Let function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula

$$f(x,y) = \begin{cases} \dfrac{\sin(xy^2)}{x^2+y^6}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is not continuous.

5. (10 pts) Let $E \subset \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}$ a function. Suppose that all partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

6. (10 pts) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = (x_1^2 - x_2)(3x_1^2 - x_2)$. Prove that f has $(0, 0)$ as a critical point but not as a local extremum.

Hint: consider $f(0, t)$ and $f(t, 2t^2)$ for t near 0.

7. (10 pts)

Let $\mu(z)$ denote the Möbius transformation which maps 1 to 0, i to 1, and -1 to ∞. What is the μ-image of the half-disk $\{z : |z| < 1, \mathrm{Im}(z) > 0\}$?

8. (10 pts) Let $f(z)$ be an entire function such that $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is of the form $f(z) = cz$, where c is a complex constant.

9. (10 pts)

Find the Laurent series of the function

$$f(z) = \frac{z}{z^2 - 1}$$

in the annulus $\{z : 0 < |z-1| < 2\}$ and in the annulus $\{z : |z-1| > 2\}$.