



1. **(10 pts)** Let $K \subset \mathbb{R}$ be a set with the following property: every continuous function $f : K \rightarrow \mathbb{R}$ is bounded. Prove that K is closed and bounded (hence compact).



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2. (10 pts) Let a_n be a sequence of positive real numbers, such that

$$\sum_{n=1}^{\infty} a_n$$

diverges. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}$$

also diverges.



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**3. (10 pts)**

Recall that the Dirichlet function $f : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

Show that the Dirichlet function is not Riemann integrable.



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4. (10 pts) Let function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by the formula

$$f(x, y) = \begin{cases} \frac{\sin(xy^2)}{x^2 + y^6}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that f is not continuous.



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5. (10 pts) Let $E \subset \mathbb{R}^n$ be an open set and $f : E \rightarrow \mathbb{R}$ a function. Suppose that all partial derivatives D_1f, \dots, D_nf are bounded in E . Prove that f is continuous in E .



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6. (10 pts) Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x_1, x_2) = (x_1^2 - x_2)(3x_1^2 - x_2)$. Prove that f has $(0, 0)$ as a critical point but not as a local extremum.

Hint: consider $f(0, t)$ and $f(t, 2t^2)$ for t near 0.



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**7. (10 pts)**

Let $\mu(z)$ denote the Möbius transformation which maps 1 to 0, i to 1, and -1 to ∞ . What is the μ -image of the half-disk $\{z : |z| < 1, \operatorname{Im}(z) > 0\}$?



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8. (10 pts) Let $f(z)$ be an entire function such that $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is of the form $f(z) = cz$, where c is a complex constant.



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**9. (10 pts)**

Find the Laurent series of the function

$$f(z) = \frac{z}{z^2 - 1}$$

in the annulus $\{z : 0 < |z - 1| < 2\}$ and in the annulus $\{z : |z - 1| > 2\}$.



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