**Analysis Qualifying Exam, January 2016**

January $24^{\text{th}}$, 2016

Last name ...............................

First name .............................

KSU Email .............................

**1. (10 pts)**

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$|f(x) - f(y)| \leq |x - y|^{\alpha},$$

for some $\alpha > 0$.

(i) Show that $f$ is uniformly continuous.

(ii) Show that if $\alpha > 1$ then $f$ must be constant.
Hint: is $f$ differentiable?

**2. (10 pts)**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$. Prove that $f$ is bounded.

**3. (10 pts)**

Show that the characteristic function of the rationals

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable over any interval $[a, b]$ in $\mathbb{R}$.

**4. (10 pts)**

Let $(f_n)$ be a sequence of functions $f_n : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, and suppose that there exist constants $M_n \geq 0$ such that

$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \qquad \text{and} \quad \sum_{n=1}^{\infty} M_n < \infty.$$

Prove that the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $A$.

**5. (10 pts)**

Describe the set of points at which the Implicit Function Theorem guarantees that the curve $x^4 + xy^6 - 3y^4 = c$ is locally the graph of a function.

**6. (10 pts)**

Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

a) Do the first partial derivatives exist at the origin?

b) Is the function differentiable at the origin?

**7. (10 pts)**

Let

$$f(z) = \frac{z}{z^2 + 6z + 8}.$$

Write the power series expansion of $f(z)$ centered at $z_0 = 0$ in the annulus $2 < |z| < 4$.

**8. (10 pts)**

Calculate using residues

$$\int_0^\infty \frac{x^3 \sin(x)}{(1+x^2)^2} dx.$$

Hint: Consider $f(z) = \frac{z^3 e^{iz}}{(1+z^2)^2}$.

**9. (10 pts)**

Find the number of roots (counting multiplicities) of the polynomial $p(z) = 3z^4 - z^3 + 8z^2 - 2z + 1$ in the annulus $\{z : 1 < |z| < 2\}$.

Hint: Use Rouché's theorem twice.