Analysis QE I

August 27, 2017

Last name .....

First name .....

Email .....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the next page for the solution. If more space is needed, use the blank pages at the back.

All pages must be submitted. If there is work to be ignored, either cross it out (or otherwise indicate its status) or tape a clean sheet over it to allow the space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt six problems. Four complete solutions will earn a passing mark. Credit for completed parts of separate problems may combine to constitute a pass as well. You may use results from one part of a problem (even if you did not solve it) in your solution to a subsequent part.

No references are to be used during the exam.

1. (10 pts) Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be Cauchy sequences in a metric space (X, d). Show that $(d(x_n, y_n))_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$.

2. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f(x) < f'(x)$ for all $x \geq 0$. Prove that $f(x) > 0$ for all $x > 0$.

3. (10 pts) Let $f(x)$ be a continuous real-valued function on $[a, b]$ such that $\int_a^b (f(x))^2 \, \mathrm{d}x = 0$. Show that $f \equiv 0$.

4. (10 pts) For $n \in \mathbb{N}$, define $f_n : [1, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{n+1}{n} e^{-nx}$. Show that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a continuous function.

5. (10 pts) Let

$$f(x,y) = \begin{cases} \frac{xy^3}{x^3+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that for every unit vector $\mathbf{u}$, the directional derivative of f in the direction $\mathbf{u}$ at the point $(0,0)$ exists.

(b) Is f continuous at $(0,0)$?

(c) Is f differentiable at $(0,0)$?

6. (10 pts) Consider the system of equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 2 \\ \sin(xyz) &= 0. \end{aligned}$$

(a) Show that there is a neighborhood of $(1, 0, 1)$ on which the solution to the system of equations can be written as $(x, y) = f(z)$, where f is a vector-valued function.

(b) Is there an $S \subset \mathbb{R}$ and a vector-valued function $f : S \to \mathbb{R}^2$ such that for all $x, y, z \in \mathbb{R}$, $(x, y) = f(z)$ iff x, y, z satisfy the system?

(c) Does the system define x and z uniquely from y in some neighborhood of $(1, 0, 1)$?

7. (10 pts) Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function ($\mathbb{D}$ is a unit disk). Is the function $\operatorname{Re} f(\bar{z})$ harmonic? Prove or give counterexample.

8. (10 pts) Let f be an entire function such that $\operatorname{Re} f > -1$. Show that f is constant. (Recall that function is entire if it is holomorphic in $\mathbb{C}$).

9. (10 pts) Use residues to calculate the integral

$$\int_0^\infty \frac{1}{1+x^4}\,\mathrm{d}x.$$