1. (10 pts)

Let (a_n) be a Cauchy sequence in a metric space (M, d). Show that if (a_n) has a convergent subsequence, then it actually converges.

2. (10 pts) Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$.

(a) Show that $\liminf_{n\to\infty} na_n = 0$.

(b) Give an example showing that $\limsup_{n\to\infty} na_n > 0$ is possible.

3. (10 pts)

Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that f takes on no value more than twice. Show that f takes on some value exactly once.

4. (10 pts)

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \left(1 - \cos \frac{x^2}{y}\right) \sqrt{x^2 + y^2} & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Calculate all the directional derivatives of f at $(0,0)$.

(c) State the definition of differentiability for a function $f : \mathbb{R}^2 \to \mathbb{R}$.

(d) Show that f is not differentiable at $(0,0)$. Hint: violate the definition.

5. (10 pts)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(u, v) = (u + v, u^2 + v^2)$.

(a) Find all points where the map is locally one-to-one. Let S be the set of these points.

(b) Is T one-to-one on S?

(c) Determine the range of T.

6. (10 pts)

Let $U = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$. Suppose that $f : U \to \mathbb{R}$ is such that both partial derivatives of f are zero at every point in U. Must f be constant? Justify your answer.

7. (10 pts)

Let a, b be given complex numbers, $|a| < |b|$. Let $|a| < r < |b|$. Calculate

$$\int_{C_r} \frac{1}{(z-a)(z-b)} dz,$$

where C_r is the circle of radius r with center 0.

8. (10 pts)

Assume that a function f is holomorphic in an open subset $U \subset \mathbb{C}$. Is the function $g = (\operatorname{Re} f)(\operatorname{Im} f)$ always harmonic in U? Prove the statement or give a counterexample.

9. (10 pts)

Let $\mathbb{D} = \{|z| < 1\}$. Does there exist a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{1}{2}) = \frac{3}{4}$, $f'(\frac{1}{2}) = \frac{2}{3}$? (Hint: use Schwarz's Lemma)