

**1. (10 pts)**

Let (a_n) be a Cauchy sequence in a metric space (M, d) . Show that if (a_n) has a convergent subsequence, then it actually converges.



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2. (10 pts) Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$.

(a) Show that $\liminf_{n \rightarrow \infty} na_n = 0$.

(b) Give an example showing that $\limsup_{n \rightarrow \infty} na_n > 0$ is possible.



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**3. (10 pts)**

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous, and suppose that f takes on no value more than twice. Show that f takes on some value exactly once.



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**4. (10 pts)**

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \left(1 - \cos \frac{x^2}{y}\right) \sqrt{x^2 + y^2} & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Show that f is continuous at $(0, 0)$.
- (b) Calculate all the directional derivatives of f at $(0, 0)$.
- (c) State the definition of differentiability for a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.
- (d) Show that f is not differentiable at $(0, 0)$. Hint: violate the definition.



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**5. (10 pts)**

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(u, v) = (u + v, u^2 + v^2)$.

- (a) Find all points where the map is locally one-to-one. Let S be the set of these points.
- (b) Is T one-to-one on S ?
- (c) Determine the range of T .



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**6. (10 pts)**

Let $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Suppose that $f : U \rightarrow \mathbb{R}$ is such that both partial derivatives of f are zero at every point in U . Must f be constant? Justify your answer.



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**7. (10 pts)**

Let a, b be given complex numbers, $|a| < |b|$. Let $|a| < r < |b|$. Calculate

$$\int_{C_r} \frac{1}{(z-a)(z-b)} dz,$$

where C_r is the circle of radius r with center 0.



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**8. (10 pts)**

Assume that a function f is holomorphic in an open subset $U \subset \mathbb{C}$. Is the function $g = (\operatorname{Re} f)(\operatorname{Im} f)$ always harmonic in U ? Prove the statement or give a counterexample.



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**9. (10 pts)**

Let $\mathbb{D} = \{|z| < 1\}$. Does there exist a holomorphic function $f : \mathbb{D} \rightarrow \mathbb{D}$ such that $f(\frac{1}{2}) = \frac{3}{4}$, $f'(\frac{1}{2}) = \frac{2}{3}$? (Hint: use Schwarz's Lemma)



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