



1. **(10 pts)** Let (M, d) be a metric space. Show that $\rho(x, y) = \sqrt{d(x, y)}$ also defines a metric. Is the identity map $i : (M, d) \rightarrow (M, \rho)$, $i(x) = x$ continuous?



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2. (10 pts) The function $f : M \rightarrow \mathbb{R}$ is called lower semicontinuous if for all $\alpha \in \mathbb{R}$ the set $\{x : f(x) > \alpha\}$ is open. Show that if f is lower semicontinuous and M is compact then
- a) f is bounded below, and
 - b) f attains a minimum value.



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**3. (10 pts)**

Let $f_n(x) = \sum_{j=1}^n \frac{1}{n} f(x + \frac{j}{n})$, where f is a continuous function on $\mathbb{R}$. Show that the sequence of functions $(f_n)_{n \in \mathbb{N}}$ converges pointwise to a continuous function.



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4. (10 pts) Suppose $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ are continuous functions.

- (a) Show that the set $B = \{x \in \mathbb{R}^n : f(x) = g(x)\}$ is closed in $\mathbb{R}^n$.
- (b) Let $p = 1$. Prove that the set $C = \{x \in \mathbb{R}^n : f(x) > g(x)\}$ is open in $\mathbb{R}^n$.



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5. (10 pts) Let the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by the formula

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

- (a) Is f continuous at $(0, 0)$?
- (b) Show that partial derivatives $D_1 f(0, 0)$ and $D_2 f(0, 0)$ exist and are equal to 0.
- (c) Is f differentiable at $(0, 0)$?



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**6. (10 pts)**

Consider the following equation for $x \in \mathbb{R}$ with $y = (y_1, y_2) \in \mathbb{R}^2$ as a parameter:

$$x^3 y_1 + x^2 y_1 y_2 + x + y_1^2 y_2 = 0. \quad (1)$$

- (a) Prove that there are neighborhoods V of $(-1, 1)$ and U of 1 such that for every $y \in V$ Eq. (??) has a unique solution $x = \psi(y)$ in U .
- (b) Find $D_1\psi(-1, 1)$ and $D_2\psi(-1, 1)$.
- (c) Prove that there do *not* exist neighborhoods V of $(-1, 1)$ and U' of -1 such that for every $y \in V$ the equation has a unique solution $x = x(y) \in U'$. *Hint:* Explicitly determine the three solutions for x in the special case where $y_1 = -1$.



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7. (10 pts) Find the number of zeroes of the function $f(z) = z^7 - 8z^2 + 2$ in the annulus $1 < |z| < 2$.



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8. (10 pts) Does there exist an entire function f such that $f\left(\frac{1}{n}\right) = \frac{n}{n+1}$?
Hint: Use the Identity Theorem.



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**9. (10 pts)**

Use the contour integral to compute $\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 4} dx$.



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