**1. (10 pts)** Let $(M, d)$ be a metric space. Show that $\rho(x, y) = \sqrt{d(x, y)}$ also defines a metric. Is the identity map $i : (M, d) \to (M, \rho)$, $i(x) = x$ continuous?

**2. (10 pts)** The function $f : M \to \mathbb{R}$ is called lower semicontinuous if for all $\alpha \in \mathbb{R}$ the set $\{x : \ f(x) > \alpha\}$ is open. Show that if $f$ is lower semicontinuous and $M$ is compact then

a) $f$ is bounded below, and

b) $f$ attains a minimum value.

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**3. (10 pts)**

Let $f_n(x) = \sum_{j=1}^{n} \frac{1}{n} f(x + \frac{j}{n})$, where $f$ is a continuous function on $\mathbb{R}$. Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a continuous function.

**4. (10 pts)** Suppose $f$, $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous functions.

(a) Show that the set $B = \{x \in \mathbb{R}^n : \ f(x) = g(x)\}$ is closed in $\mathbb{R}^n$.

(b) Let $p = 1$. Prove that the set $C = \{x \in \mathbb{R}^n : \ f(x) > g(x)\}$ is open in $\mathbb{R}^n$.

**5. (10 pts)** Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Is $f$ continuous at $(0,0)$?

(b) Show that partial derivatives $D_1 f(0,0)$ and $D_2 f(0,0)$ exist and are equal to 0.

(c) Is $f$ differentiable at $(0,0)$?

**6. (10 pts)**

Consider the following equation for $x \in \mathbb{R}$ with $y = (y_1, y_2) \in \mathbb{R}^2$ as a parameter:

$$x^3 y_1 + x^2 y_1 y_2 + x + y_1^2 y_2 = 0. \tag{1}$$

(a) Prove that there are neighborhoods $V$ of $(-1, 1)$ and $U$ of 1 such that for every $y \in V$ Eq. (**??**) has a unique solution $x = \psi(y)$ in $U$.

(b) Find $D_1\psi(-1, 1)$ and $D_2\psi(-1, 1)$.

(c) Prove that there do *not* exist neighborhoods $V$ of $(-1, 1)$ and $U'$ of $-1$ such that for every $y \in V$ the equation has a unique solution $x = x(y) \in U'$. *Hint:* Explicitly determine the three solutions for $x$ in the special case where $y_1 = -1$.

**7. (10 pts)** Find the number of zeroes of the function $f(z) = z^7 - 8z^2 + 2$ in the annulus $1 < |z| < 2$.

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**8. (10 pts)** Does there exist an entire function $f$ such that $f\left(\frac{1}{n}\right) = \frac{n}{n+1}$?
*Hint:* Use the Identity Theorem.

**9. (10 pts)**

Use the contour integral to compute $\displaystyle\int_0^\infty \frac{x^2}{x^4+5x^2+4}\,dx$.