Analysis QE I Exam
June 2018

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt six problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts) Let (X, d) be a metric space. Let $f : X \to X$ be a continuous map. Assume that for all $x, y \in X$,

$$d(f(x), f(y)) < d(x, y).$$

a) Show that f has at most one fixed point.

b) Show that if X is compact, f has exactly one fixed point.

2. (10 pts) Let $K > 0$. The function $f : [a, b] \to \mathbb{R}$ is K-Lipschitz if for all $x, y \in [a, b]$:

$$|f(x) - f(y)| \leq K|x - y|$$

a) Assume that f has a bounded derivative on (a, b). Show that there exists K such that f is K-Lipschitz.

b) For every K, give an example of the function that is K-Lipschitz, but not differentiable.

3. (10 pts) Prove or disprove:

a) The product of two uniformly continuous functions on $\mathbb{R}$ is also uniformly continuous.

b) The product of two uniformly continuous functions on $[0, 1]$ is also uniformly continuous.

4. (10 pts) Let I be a rectangle in $\mathbb{R}^2$ and suppose f is continuous on I. Prove that there exists a point $x_0 \in I$ such that

$$\int_I f(x)\,dx = f(x_0)\,\mathrm{vol}(I),$$

where $\mathrm{vol}(I)$ is the n-dimensional volume of the rectangle.

5. (10 pts)

Let function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Is f continuous at $(0,0)$?

(b) Show that both partial derivatives $D_1 f(0,0)$ and $D_2 f(0,0)$ exist and compute them.

(c) Is f differentiable at $(0,0)$?

6. (10 pts)

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the formula

$$F(x_1, x_2) = e^{x_1}(\cos(x_2), \sin(x_2)).$$

(a) Find the image of F.

(b) Prove that for every $x \in \mathbb{R}^2$ there exists a neighborhood U in $\mathbb{R}^2$ such that $F : U \to F(U)$ is a diffeomorphism, but that F is not injective on all of $\mathbb{R}^2$.

(c) Let $x = (0, \pi/3)$, $y = F(x)$ and let H be the continuous inverse of F, defined in a neighborhood of y, such that $H(y) = x$. Give an explicit formula for H.

7. **(10 pts)** Calculate the integral $\int_C \frac{\cos z}{z^3+4z}\, dz$, where C is counter-clockwise oriented circle of radius 2 with center at the point $z = i$.

8. (10 pts)

Let $\mathbb{D} = \{|z| < 1\}$. Consider the set of holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{3}{4}) = 0$. What are the possible values of $f'(\frac{3}{4})$?

9. (10 pts)

Let $f(z)$ be an entire function that does not take negative real values. Show that f is constant. (Hint: consider $\sqrt{f}$).