Analysis QE I Exam
August 2019

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

Prove the L^1 Chebyshev inequality: for any real $s > 0$,

$$|\{x : |f(x)| > s\}| \leq \frac{1}{s} \int |f|.$$

2. (10 pts) What is the Lebesgue measure of the set of rationals in the line? Give a proof of your assertion.

3. (10 pts) Suppose $\{f_n,\ n \geq 1\}$ is a family of real-valued functions on a compact interval I that are Hölder continuous with exponent α and constant M: i.e., for all $n \geq 1$ and all $x, y \in I$,

$$|f_n(x) - f_n(y)| \leq M|x - y|^\alpha.$$

Suppose also that the set $\{f_n(x_0) \mid n \geq 1\}$ is bounded for some fixed $x_0 \in I$. Prove that $(f_n)_{n=1}^\infty$ has a subsequence converging uniformly to a function f that is Hölder continuous with the same exponent α and constant M.

4. (10 pts)

Let $U \in \mathbb{R}^n$ be an open set, $a \in U$, and $f : U \to \mathbb{R}^m$. Prove that the following statements are equivalent:

(a) The mapping f is differentiable at a.

(b) Every component function $f_i : U \to \mathbb{R}$ of f, $1 \leq i \leq m$, is differentiable at a.

5. (10 pts)

Let f be an entire function such that $f(z) = f(z + 2\pi)$ and $f(z) = f(z + 2\pi i)$ for all $z \in \mathbb{C}$. Prove that f is constant.

6. (10 pts)

Assume a and b are complex with $|a| \neq 1$. Evaluate, distinguishing cases.

$$\int_\gamma \left(\frac{z-b}{z-a} \right)^2 dz,$$

where $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.