Analysis QE I Exam
June 2019

Name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. (10 pts)

Assume a function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree 1, in the sense that $f(tx) = tf(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

(a) Show that f has directional derivatives at 0 in all directions.

(b) Prove that f is differentiable at 0 if and only if f is linear.

2. (10 pts)

Find all functions f that are holomorphic in the disk $D(0;1)$ and such that

$$f(1/n) = n^2(f(1/n))^3, \quad \text{for } n = 2, 3, 4, \ldots.$$

3. (10 pts) Let f be an entire function. Prove that if $f(z)$ is real for all z with $|z| = 1$, then f is constant.

4. (10 pts)

Prove that the family of all polynomials $P(x)$ of degree $\leq N$ with coefficients in $[-1, 1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.

5. (10 pts) What is the Lebesgue measure of the Cantor set?

6. (10 pts) Prove that a non-negative measurable function has integral equal to zero if and only if it is zero almost everywhere.