Analysis QE I Fall 2020

**1.** Show that both series

$$\sum_{n=1}^{\infty} x^n(1-x) \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^n x^n(1-x)$$

are convergent on $[0,1]$, but only one converges uniformly. Which one? Why?

**2.** Let $f:\mathbb{R}\to\mathbb{R}$. Show that each of the following conditions implies that $f$ is Borel measurable:

(a) $f$ is increasing;

(b) $f$ is lower semi-continuous, i.e. $f(x) \leq \liminf_{y\to x} f(y)$, for all $x\in\mathbb{R}$.

**3.** Let $f:\mathbb{R}^2\to\mathbb{R}$ be a continuously differentiable function. Show that $f$ is not injective.

**4.** Let $f(z)$ and $g(z)$ be two entire functions, such that

$$|f(z)| \leq |g(z)|, \quad z\in\mathbb{C}.$$

Prove that there exists a constant $c\in\mathbb{C}$, such that

$$f(z) = cg(z), \quad z\in\mathbb{C}.$$

**5.** How many distinct roots does the polynomial

$$p(z) = z^7 + 10z^4 + 7$$

have in the disk $|z|\leq 1$?

**6.** Use residues to compute

$$\int_0^{\infty} \frac{x\sin(2x)}{4+x^2}dx.$$