Analysis QE I Exam June 2020

Instructions: Do not write your name on any of the pages. This is a closed book exam: no books, no notes, no calculators etc. Only plain papers and pens should be on your table.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

   (a) Is f differentiable at $(0,0)$?

   (b) Are the partial derivatives $D_i f$, $i = 1,2$ continuous at $(0,0)$?

2. Let the sequence of functions be defined by $f_n(x) = nxe^{-nx}$ on $[0,+\infty)$. Determine the pointwise limit on the given interval (if it exists) and an interval on which the convergence is uniform (if any).

   Does the sequence of derivatives (f_n') converge uniformly on $[0,+\infty)$?

3. Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$ is measurable. Show that f is measurable if and only if the function $g : \mathbb{R} \to \mathbb{R}$ is measurable, where $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ otherwise.

4. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be given by

$$f(z) = z|z|.$$

   Where is $f'(z)$ defined? Where is $f(z)$ analytic?

5. Construct a map that maps the half-strip

$$S = \{z : |\mathfrak{Re}(z)| < 1, \mathfrak{Im}(z) > 0\}$$

   conformally onto the open unit disk

$$\mathbb{D} = \{z : |z| < 1\}.$$

6. Let $f(z)$ be analytic in the punctured disk

$$D = \{z : |z| < 1, z \neq 1/2\}.$$

   Suppose that $f(z)$ has a simple pole at $z = 1/2$ and that

$$\operatorname*{Res}_{z=1/2} f(z) = 1.$$

   Determine the coefficient a_{-2} in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad 1/2 < |z| < 1.$$