Analysis QE I August 2021

1. Let $a < b$ be two points on the real line and let $f(x)$ be a function, thrice differentiable on the interval $[a, b]$. Prove that there is a point c between a and b, such that
$$\frac{f(b)-f(a)}{b-a} = \frac{f'(a)+f'(b)}{2} - f'''(c)\frac{(b-a)^2}{12}.$$
Hint: Consider the auxiliary function
$$g(x) = f(x) - f(a) - (x-a)\frac{f'(a)+f'(x)}{2} - k(x-a)^3,$$
where k is a suitable constant.

2. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by
$$f(x,y) = x^4 + x^2y^2 + xy^3 + y^4.$$
Let $S \subset \mathbb{R}^2$ be the set of solutions of the equation $f(x,y) = 1$. Prove that every point in S has a neighborhood, where the equation can be solved for x in terms of y or vice versa.

3. Let (X, d) be a compact metric space, and let $\mathcal{F}$ be a family of real-valued functions on X. Assume that the family $\mathcal{F}$ is *pointwise* equicontinuous: for every $x \in X$ and for every $\epsilon > 0$ there is $\delta > 0$, such that for every function f from the family $\mathcal{F}$ it holds that
$$|f(x) - f(y)| < \epsilon,$$
whenever $y \in X$ is such that $d(x,y) < \delta$.

Prove that the family $\mathcal{F}$ is *uniformly* equicontinuous: for every $\epsilon > 0$ there is $\delta > 0$, such that for every function f from the family $\mathcal{F}$ it holds that
$$|f(x) - f(y)| < \epsilon,$$
whenever $x, y \in X$ are such that $d(x,y) < \delta$.

4. Consider the open unit disk $\mathbb{D} = \left\{z \in \mathbb{C} \ : \ |z| < 1\right\}$ and let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function satisfying $f(0) = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = 0$. Prove that $f(z) = \dfrac{2z-1}{z-2}$, $\forall\, z \in \mathbb{D}$.

5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, and for every positive integer n, there exist a positive constant C_n and a neighborhood V_n of 0, such that
$$|f(z)| \le C_n|z|^n, \ \ \forall\, z \in V_n.$$
Prove that f is the constant zero function.

6. Let Γ be the circle of radius 2 centered at i, parametrized counterclockwise. Compute the complex line integral
$$\oint_\Gamma \frac{\sin(\pi z)}{z^4 + 3z^3 + 2z^2}dz.$$