Analysis QE I June 2022

INSTRUCTIONS

This is a closed-book exam. No written material, electronic tools, or communication with others is permitted. You have three hours to work and 20 minutes to upload solutions.

Four questions correctly solved (up to minor errors) will earn a pass on this exam. Parts of questions may in some cases combine to count as one full problem.

Do not write your name or any other identifying information on your work.

Submitting your work:

- Find the email from Crowdmark Mailer with subject "Graduate Program 2021–2022 – New Assignment: Analysis" in the subject line. Follow steps given.
- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to sarahrez@ksu.edu.
- Before leaving the library, give all hard copies to the proctor.

(1) Let (X, Σ, μ) be a measure space, and let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of measurable functions. Prove that the set of points in X, where sequence $f_n(x)$ converges, is measurable.

(2) Let (X, d) be a compact metric space, and let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of continuous functions, such that series

$$\sum_{n=1}^{\infty} |f_n(x)|$$

converges (pointwise) to a continuous function on X. Prove that series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges to a continuous function on X, as well.

(3) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuously differentiable. Prove, using the Implicit Function Theorem, that f cannot be 1-to-1.

(4) Calculate, using residues:

$$\int_0^{\infty} \frac{\cos(x)}{1 + x^4}\, dx.$$

(5) Is there a function $f(z)$, analytic in the disk $|z| < 1$ and such that

$$f\left(\frac{1}{n+1}\right) = \frac{1}{\sqrt{n}}, \quad \text{for all } n \in \mathbb{N}?$$

(6) How many zeros (counting multiplicities) does polynomial

$$p(z) = z^4 + 4iz^3 + 7z + 1$$

have in the annulus $1 < |z| < 2$?