Analysis QE I Exam
August 2023

Name . . . .

KSU Email . . . .

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. **(10 pts)**

   (a) Give a definition of a compact space. Let $f : X \to Y$ be a function between metric spaces. Give a definition of f being continuous and uniformly continuous.

   (b) Is it true that a continuous function $f : X \to \mathbb{R}$ is uniformly continuous if X is compact? Justify you answer by providing a proof or an example.

2. **(10 pts)** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers, such that $\sum_{n=1}^{\infty} x_n < \infty$.

   (a) Prove that $\liminf_{n\to\infty} nx_n = 0$.

   (b) Does $\lim_{n\to\infty} nx_n$ always exist? Justify your answer.

3. **(10 pts)** Define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$f(x, y) = (e^{3x} \cos(2y), e^{3x} \sin(2y)).$$

(a) Give a definition of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ being differentiable and continously differentiable in $\mathbb{R}^2$.

(b) Compute the derivative $D_f(x, y)$ and the Jacobian determinant $J_f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Is f continuously differentiable at every $(x, y) \in \mathbb{R}^2$? Is $D_f(x, y)$ invertible at every $(x, y) \in \mathbb{R}^2$.

(c) Is f injective? Is f surjective? Explain.

4. **(10 pts)** Suppose f is analytic on $\Omega = \{z : |z| > 2\}$ and maps Ω into itself

   (a) Could such an f have an essential singularity at infinity? Explain.

   (b) If $\lim_{z\to\infty} f(z) = \infty$, prove that $|f(z)| \geq |z|$ for all z in Ω.

5. **(10 pts)** Does there exist an entire function f such that

$$f\left(\frac{1}{n}\right) = \frac{n^2}{1+n^3}$$

for $n = 1, 2, \ldots$? Justify your answer by providing a proof.

6. **(10 pts)** Compute the integral

$$\int_0^\infty \frac{dx}{(x^2+9)(x^2+4)}.$$