Analysis QE I Exam
June 2023

Name . . . . . .

KSU Email . . . . . .

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the back of the page and the next page for the solution. If more space is needed, use the blank pages at the end.

All pages must be submitted. If there is work you want ignored, cross it out (or otherwise indicate its status) or tape a clean sheet over it to create more space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a pass. Credit for completed parts of separate problems may combine to result in a pass.

No references are to be used during the exam.

1. **(10 pts)** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers. Show that if $\sum_{n=1}^{\infty} x_n$ diverges then

$$\sum_{n=1}^{\infty} \frac{x_n}{5+\sqrt{x_n}} = +\infty.$$

2. **(10 pts)**

(a) Give a definition of an equicontinuous family of functions on a metric space.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given. Assume that sequence $f_n(x) = f(nx)$ is equicontinuous. What can you say about $f(x)$?

3. **(10 pts)** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Is f is continuous at $(0,0)$?

(b) Find partial derivatives of f at $(0,0)$.

(c) Is f differentiable at $(0,0)$?

4. **(10 pts)** Suppose that f is analytic on the upper half plane and maps the upper half plane into the unit disc. Prove that $|f'(i)| \leq \frac{1}{2}$. Describe all f for which $|f'(i)| = \frac{1}{2}$.

5. **(10 pts)** Prove that if P is holomorphic on $\mathbb{C}$ and $\lim_{z\to\infty} P(z) = \infty$, then P is a polynomial.

6. **(10 pts)** Construct a conformal map between a quarter disk

$$Q = \{z : |z| < 1, \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$$

and the unit disk

$$\mathbb{D} = \{z : |z| < 1\}.$$