Analysis QE1

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Instructions: You have three hours to do your work and fifteen minutes to upload the solutions.

You must **show your work clearly and justify everything** to receive credit.

0) Four full problems correctly done will earn a pass. Parts of several problems may combine to count for a full problem.
1) Do not write your name on any of the pages.
2) This is a closed book exam: no books, no notes, no calculators etc. Only plain papers and pens should be on your table.
3) You must have your camera on showing your hands and, if possible, at least part of your faces during the exam.
4) After you receive the exam online by email, if you want you can print it or you can keep it open on your laptop or cellphone. After that you are allowed to use your computer or any electronic device during the exam only to read the exam and later to upload it on Crowdmark. Also you can communicate to the examiner via private chat in Zoom.
5) In case you lose internet connection at some point, you can continue your exam, however the examiners might consider to have an oral reexamination with you where you would need to explain steps in your work. You can be asked additional questions. In case the loss of connection is long, you can contact the examiner.
6) The exam is supposed to take 3 hours not counting the time of printing or accessing the problems and uploading your test. If you want you can have a bit of extra time, but the exam must be uploaded to Crowdmark no less than 10 minutes after the exam

Submitting your work:

- Each photo must have work from only one problem: if you have multiple problems on one page, use blank pages to cover other work or crop your photos accordingly.
- If you have trouble uploading, please send photos of individual problems to tinaande@ksu.edu.

Problem 1 [10 points] Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(i) Define what it means for a sequence to be bounded.

(ii) Define what it means for a sequence to be Cauchy.

(iii) Prove that Cauchy sequences are bounded.

Problem 2 [10 points] Let (X, d_X) be a metric space.

(i) Define what it means for a metric space to be compact using open covers.

(ii) Define what it means for a function $f : X \to \mathbb{R}$ to be continuous at a point $x_0 \in X$.

(iii) Define what it means for a function $f : X \to \mathbb{R}$ to be uniformly continuous on X.

(iv) Finally, assuming that (X, d_X) is compact and $f : X \to \mathbb{R}$ is a continuous function, show that f is necessarily uniformly continuous.

Problem 3 [10 points] Let $\Omega \subset \mathbb{R}^N$ be measurable and let $f_j : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ for $j = 1, 2, 3, \ldots$ be non-negative measurable functions.

(i) Write the definition of $\liminf_{j\to\infty} f_j$ using inf's and sup's.

(ii) State the Monotone Convergence Theorem

(iii) Use part (ii) to prove Fatou's Lemma:

$$\int_\Omega \liminf_{j\to\infty} f_j \leq \liminf_{j\to\infty} \int_\Omega f_j.$$

Problem 4 [10 points] Let D be a convex complex domain, and let $f : D \longrightarrow \mathbb{C}$ be continuous. Assume that for every closed triangular region $\Delta \subset D$ with the counterclockwise oriented boundary δ it holds that

$$\int_{\delta} f(z)dz = 0.$$

Prove that there exists an analytic function $F : D \longrightarrow \mathbb{C}$, such that

$$F'(z) = f(z) \quad \forall z \in D.$$

Problem 5 [10 points] Let D be a complex domain, and let $f : D \longrightarrow \mathbb{C}$ be a non-constant analytic function. Assume that $z_0 \in D$ is a point of local minimum for the function $|f(z)|$. Prove that $f(z_0) = 0$.

Problem 6 [10 points] Calculate using residues:

$$\oint_{|z|=3} \frac{\tan(z)}{z^2} dz.$$