Analysis QE I Sample Exam

Attempt six problems. You must solve four problems to pass. Justify your answers. Please use a separate sheet of paper for each problem.

1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals that converges to 0. Prove that there is a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges absolutely.

2. Show that if f is a nonnegative continuous function defined on $[0,1]$ satisfying $\int_0^1 f(x)\,\mathrm{d}x = 0$, then $f \equiv 0$ on $[0,1]$.

3. (a) Let K be a compact subset of $\mathbb{R}$ and let $f : K \to \mathbb{R}$ be continuous. Show that f attains its maximum value: i.e., there is a point $a \in K$ such that $\forall x \in K$, $f(x) \le f(a)$.

   (b) Suppose $f : \mathbb{R} \to (0,\infty)$ is a continuous function with $\lim\limits_{x\to\pm\infty} f(x) = 0$. Show that f attains its maximum value.

4. Show that the system of equations (note that these are not linear)

$$\begin{aligned} 3x + 7 - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

   cannot be solved for x, y, z in terms of u but can be solved for each of the other sets of three variables in terms of the remaining one.

5. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$.

   - Show that the converges pointwise on $[0,\infty)$. To what function?
   - Does the series converge uniformly on $[0,1]$? On $[1,\infty)$?

6. Does there exist a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$\int_{-\pi}^{\pi} x f(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{-\pi}^{\pi} x^n f(x)\,\mathrm{d}x = 0$$

   for $n = 0, 2, 3, 4, \ldots$? Give an example or prove that no such f exists. Hint: calculate the Fourier coefficients of f using the power series expansion for e^x.

7. Compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x\,.$

8. Suppose that f is a complex-valued analytic function in the open unit disk $\mathbb{D}$ such that $|f|$ is constant. Prove that f is constant.

9. Find a conformal map from the strip $\{z \in \mathbb{C} \,:\, |\mathrm{Re}(z)| < 1\}$ onto the open disk $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$.