Applied Mathematics Qualifying Exam, August 2015

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, let $W \in \mathbb{R}^{n\times m}$, and define

$$B := W^T A W.$$

(a) Show that B is symmetric positive semi-definite.
(b) Find conditions on W that are necessary and sufficient for B to be positive definite.

Problem 2. Let

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and let $x_0 = \frac{1}{2}(1,1,1,1)^T$. For $k = 1, 2, 3, \ldots$, define

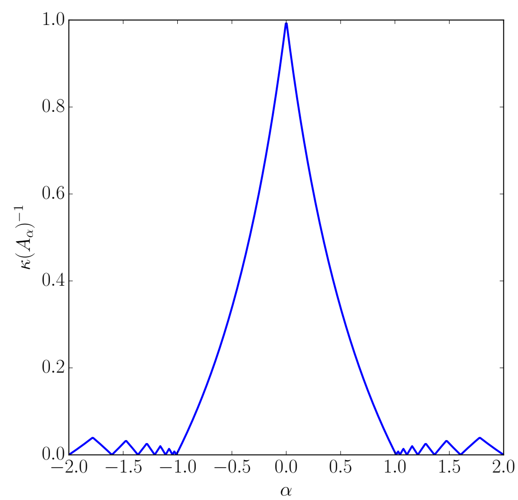
$$y_k = Ax_{k-1}, \quad x_k = \frac{y_k}{\|y_k\|}, \quad \eta_k = x_k^T A x_k.$$

Determine the value of $\lim_{k\to\infty} \eta_k$, if it exists. Explain your reasoning.

Problem 3. Consider the family of matrices $A_\alpha \in \mathbb{R}^{n\times n}$ of the form

$$A_\alpha = \begin{bmatrix} 2 & \alpha & & & & \\ \alpha & 2 & \alpha & & & \\ & \alpha & 2 & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 2 & \alpha \\ & & & & \alpha & 2 \end{bmatrix}.$$

The figure below shows the reciprocal of the condition number $\kappa(A_\alpha)^{-1}$ plotted as a function of α for $n = 20$. Prove that, for any choice of n, $\kappa(A_\alpha)$ is finite for $|\alpha| < 1$. (Hint: Use Gershgorin's Theorem.)



Problem 4. Consider the differential equation:

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) - 2y = f(x), \quad x \in [1,2],$$

with the boundary conditions:

$$y(1) = 0, \quad y(2) = 0.$$

(a) Write the equation in the form $Ly = f$ and show that the operator L is self-adjoint with the given boundary conditions (in $L^2(-1,1)$ with the usual inner product).
(b) Find the Green's function for the equation.
Hint: Remember that the solutions to the Euler equation $x^2y'' + axy' + by = 0$ can be found in the form $y = x^r$.

Problem 5. Consider the linear integral equation of the form:

$$y(x) - \lambda \int_a^b g(x)h(t)y(t)dt = f(x), \quad x \in [a,b],$$

where $g,\ h,\ f \in L^2(a,b)$.
(a) Derive the necessary and sufficient conditions on $\lambda \in \mathbb{R}$ and $f \in L^2(a,b)$ for solvability of this equation.
(b) Solve the integral equation in the particular case when $a = 0,\ b = \pi,\ g(x) = \sin x$, and $h(t) = \sin t$.

Problem 6. Let $u \in L^2(-\pi,\pi)$ be defined as

$$u(x) = \begin{cases} \cos(x) & \text{if } |x| < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Find D^2u, the second distributional derivative of u.