Applied Mathematics Qualifying Exam, June 2015

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ be the tridiagonal matrix of the form

$$A = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

(a) Show that A has a Cholesky decomposition $A = LL^T$ with L of the form

$$L = \begin{bmatrix} 1 & & & & \\ \alpha & 1 & & & \\ & \alpha & 1 & & \\ & & \ddots & \ddots & \\ & & & \alpha & 1 \end{bmatrix}.$$

(b) Describe an algorithm for solving the linear system $Ax = b$ using only $2n - 2$ floating point additions and no other floating point operations.

Problem 2.

(a) Prove that all eigenvalues of a real symmetric matrix are real.

(b) Prove that all eigenvalues of the tridiagonal matrix

$$A = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}$$

are strictly positive.

Problem 3. Consider the matrix

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -3 & 4 \\ -1 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

(a) Compute the singular value decomposition of A. (Hint: Notice that $A = BC^T$ with B^TB and C^TC both diagonal.)

(b) Find the best rank-one approximation to A in the operator 2-norm.

Problem 4. Consider the differential equation

$$\frac{d}{dx}\big((1+x)u'(x)\big) = f(x), \qquad u(0) = u'(1) = 0.$$

(a) Write the equation in the form $Lu = f$ and compute the adjoint operator L^* (in $L^2(0,1)$ with the usual inner product).

(b) Verify that

$$g(x,y) = -\log\left(1 + \min(y,x)\right)$$

is the Green's function for the differential equation.

Problem 5. Let $k(x,y) = 2xy^2$ and define the operator K on $L^2(0,1)$ as

$$(Ku)(x) = \int_0^1 k(x,y)u(y)\,dy.$$

Given $\lambda \in \mathbb{R}$, define $L_\lambda = I - \lambda K$.

(a) Derive necessary and sufficient conditions on $\lambda \in \mathbb{R}$ and $f \in L^2(0,1)$ for the solvability of the equation

$$L_\lambda u = f.$$

(b) Construct the resolvent operator, R_λ, when it exists.

Problem 6. Let t be the distribution defined on any test function ϕ by

$$\langle t, \phi\rangle = \int_{-\infty}^{\infty} e^{-|x|}\phi(x)\,dx.$$

(a) Derive an explicit formula for the second distributional derivative $D^2 t$ of t.

(b) Find a differential equation on $\mathbb{R}$ that t solves weakly.