Applied Math Qualifying Exam, August 2016

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where h is the space between grid points, I is the identity matrix, and T is the matrix

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(a) Show that L is negative semidefinite.

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Let A be the product

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