**Applied Math Qualifying Exam, January 2016**

2016-06-04

Last name ..............................

First name .............................

KSU Email .............................

**Instructions**: Use the space below the statement of a problem as well as the next page for the solution. If more space is needed, use the blank pages at the back.

All pages must be submitted. If there is work to be ignored, either cross it out (or otherwise indicate its status) or tape a clean sheet over it to allow the space to be used, being careful not to cover the code at the top. Normally four complete solutions constitutes a passing score; note that the fifth problem here counts as two.

No references are to be used during the exam.

**1. (10 pts)**

Let $A$, $B$ be symmetric $n \times n$ matrices. Prove that $AB$ has real eigenvalues.

**2.** **(10 pts)** Consider the system $Ax = b$, where $A$ is a non-singular $n \times n$ matrix, $x, b \in \mathbb{R}^n$. Consider an iterative scheme of solution of this system:

$$x_{k+1} = (I - A)x_k + b, \quad k = 1, 2, 3, \ldots,$$

where the initial approximation $x_0$ is chosen arbitrarily.

Assume that $||I - A||_F < 1$ where $|| \cdot ||_F$ denotes the Frobenius norm. Prove that this iterative scheme converges for any initial approximation $x_0$.

**3. (10 pts)**

Consider a matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Find a $QR$ factorization of the matrix $A$;

(b) Use the $QR$ factorization to solve the system $Ax = b$, where $b = [-1, 1, 0]^T$.

**4. (10 pts)**

The function $(1/x) : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is not locally integrable and therefore does not define a regular distribution. Define its principal value by

$$\text{p.v.}\frac{1}{x}(\phi) = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x}\, dx,$$

for all $\phi \in C_c^\infty(\mathbb{R})$. Show that the limit exists and p.v.$(1/x)$ is a distribution.

**5. (20 pts)**

Let $K : L^2(0,1) \to L^2(0,1)$ be the following linear integral operator:

$$Ku(x) = \int_0^1 x\, y\, u(y)\, dy.$$

(a) Show that $K$ is a self-adjoint compact operator.

(b) Find the eigenvalues and eigenfunctions of $K$.

(c) Consider the integral equation

$$u - \mu K u = f,$$

where $\mu \neq 0$ is a real number. Derive necessary and sufficient conditions on $\mu$ and $f \in L^2(0,1)$ for the existence and uniqueness of solutions for this equation.