Applied Math Qualifying Exam, August 2017

August 26, 2017

Last name ....

First name ....

KSU Email ....

Instructions:

Do not write your name or any other identifying information on any page except this cover page.

Use the space below the statement of a problem as well as the next page for the solution. If more space is needed, use the blank pages at the back.

All pages must be submitted. If there is work to be ignored, either cross it out (or otherwise indicate its status) or tape a clean sheet over it to allow the space to be used, being careful not to cover the code at the top.

You have three hours to work on these problems. Attempt all problems. Four complete solutions will earn a passing mark. Credit for completed parts of separate problems may combine to constitute a pass as well. You may use results from one part of a problem (even if you did not solve it) in your solution to a subsequent part.

No references are to be used during the exam.

1. (10 pts)

Let A be an $n \times n$ nonsingular matrix. Show that:

$$\min \left\{ \frac{||\delta A||_2}{||A||_2} : \, A + \delta A \text{ is singular} \right\} = \frac{1}{\kappa_2(A)},$$

where $\kappa_2(A) = ||A||_2 ||A^{-1}||_2$ is the condition number of the matrix A with respect to the matrix operator norm $|| \cdot ||_2$ corresponding to the Euclidean norm on the space $\mathbb{R}^n$.

2. (10 pts)

Let $S \in \mathbb{C}^{n \times n}$ be a skew-Hermitian, i.e $S^* = -S$. Show the following:

(1) The eigenvalues of S are purely imaginary.

(2) The matrix $I - S$ is invertible.

3. (10 pts)

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

(1) Using the Gram-Schmidt algorithm, find the QR-factorization of the matrix A.

(2) Use the QR factorization from step (1) to solve the system of linear equations $Ax = b$ with $b = [2, 0, 1]^T$.

(3) Find the matrix of an orthogonal projection on the linear subspace of $\mathbb{R}^3$ spanned by the first two columns of the matrix A. (Hint: the matrix of an orthogonal projection on a plane in $\mathbb{R}^3$ can be found by using the formula $P = I - nn^T$ where n is a unit normal to the plane written as a column vector).

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4. (10 pts)

Suppose $f(x)$ and $g(x)$ are 2π periodic functions with Fourier series representations

$$f(x) = \sum_{k=-\infty}^{\infty} f_k \, e^{ikx}, \qquad g(x) = \sum_{k=-\infty}^{\infty} g_k \, e^{ikx}.$$

Find the Fourier representation of

$$h(x) = \int_0^{2\pi} f(x-y) \, g(y) \, dy.$$

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5. (10 pts)

Let

$$w(x) = \begin{cases} 3x & , \quad x > -1 \\ x^2 & , \quad x \leq -1 \end{cases} .$$

Find the distributional derivative of w.

6. (10 pts)

Consider the integral equation

$$u(x) - \lambda \int_0^x x\, u(y)\, dy = f(x),$$

where $\lambda \neq 0$ is a real number.

Derive necessary and sufficient conditions on λ and $f \in L^2(0,1)$ for the existence of solutions for this equation.

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