



Applied Math Qualifying Exam, June 2017

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Last name

First name

KSU Email



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**1. (10 pts)**

Consider the operator norm of a square $n \times n$ matrix A :

$$\|A\|_p = \max_{\mathbf{x} \neq 0, \mathbf{x} \in \mathbb{R}^n} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p},$$

where

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}},$$

and $1 \leq p < \infty$.

(1) Prove that for a non-singular matrix A , the condition number $\kappa(A) = \|A\|_p \cdot \|A^{-1}\|_p$ satisfies the inequality $\kappa(A) \geq 1$.

(2) Compute $\kappa(Q)$ where Q is an orthogonal square matrix if $\|\cdot\|_2$ norm is used.



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**2. (10 pts)**

Let $u, v \in \mathbb{R}^n$ be column vectors and define $A = I - uv^T$.

- (1) Find the determinant of the matrix A .
- (2) Show that if A is non-singular then its inverse can be written in the form $A^{-1} = I + \alpha uv^T$ and compute α .



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**3. (10 pts)**

Find a singular value decomposition of the matrix A and compute the matrix of rank 1 which is the closest in $\|\cdot\|_2$ norm to the matrix A :

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$



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**4. (10 pts)**

Use the matrix exponential method to solve the following system of first order differential equations:

$$\begin{cases} \frac{dx_1}{dt} = x_1 + x_2 + x_3 \\ \frac{dx_2}{dt} = x_1 + 2x_2 \\ \frac{dx_3}{dt} = x_1 + 2x_3 \end{cases}$$

with initial conditions $x_1(0) = 3$, $x_2(0) = 6$, $x_3(0) = 0$.



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**5. (10 pts)**

Let $H = L^2[-1, 1]$ with the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$ and let $S = \{1, x, x^2, x^3, \dots\} \subset H$.

(a) Generate an orthogonal set Q from S by applying the Gram-Schmidt procedure.

(b) Using the fact that S is a complete set in H , show that the orthogonal set Q in part (a) is complete in H .



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**6. (10 pts)**

(a) Discuss existence and uniqueness of solutions $u \in L^2[-\pi, \pi]$ for the equation

$$u(x) - \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \cos y u(y) dy = \sin 2x.$$

(b) Let K be the following operator:

$$Ku(x) = \int_{-\pi}^{\pi} \cos x \cos y u(y) dy, \quad u \in L^2[-\pi, \pi].$$

Find the eigenvalues and eigenfunctions of K .



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