Applied Math Qualifying Exam, June 2017

2017-06-08

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1. (10 pts)

Consider the operator norm of a square $n \times n$ matrix A:

$$||A||_p = \max_{\mathbf{x}\neq 0,\ \mathbf{x}\in\mathbb{R}^n} \frac{||A\mathbf{x}||_p}{||\mathbf{x}||_p},$$

where

$$||\mathbf{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}},$$

and $1 \leq p < \infty$.

(1) Prove that for a non-singular matrix A, the condition number $\kappa(A) = ||A||_p \cdot ||A^{-1}||_p$ satisfies the inequality $\kappa(A) \geq 1$.

(2) Compute $\kappa(Q)$ where Q is an orthogonal square matrix if $||\cdot||_2$ norm is used.

2. (10 pts)

Let $u, v \in \mathbb{R}^n$ be column vectors and define $A = I - uv^T$.

(1) Find the determinant of the matrix A.

(2) Show that if A is non-singular then its inverse can be written in the form $A^{-1} = I + \alpha uv^T$ and compute α.

3. (10 pts)

Find a singular value decomposition of the matrix A and compute the matrix of rank 1 which is the closest in $||\cdot||_2$ norm to the matrix A:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

4. (10 pts)

Use the matrix exponential method to solve the following system of first order differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} &= x_1 + x_2 + x_3 \\ \\ \dfrac{dx_2}{dt} &= x_1 + 2x_2 \\ \\ \dfrac{dx_3}{dt} &= x_1 + 2x_3 \end{cases}$$

with initial conditions $x_1(0) = 3,\ x_2(0) = 6,\ x_3(0) = 0.$

5. (10 pts)

Let $H = L^2[-1,1]$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ and let $S = \{1, x, x^2, x^3, \ldots\} \subset H$.

(a) Generate an orthogonal set Q from S by applying the Gram-Schmidt procedure.

(b) Using the fact that S is a complete set in H, show that the orthogonal set Q in part (a) is complete in H.

6. (10 pts)

(a) Discuss existence and uniqueness of solutions $u \in L^2[-\pi, \pi]$ for the equation

$$u(x) - \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \;\cos y \; u(y) \; dy = \sin 2x.$$

(b) Let K be the following operator:

$$Ku(x) = \int_{-\pi}^{\pi} \cos x \;\cos y \; u(y) \; dy, \quad u \in L^2[-\pi, \pi].$$

Find the eigenvalues and eigenfunctions of K.